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A MODIFICATION OF SHELAH'S ORACLE-C.C. WITH APPLICATIONS

WINFRIED JUST

ABSTRACT. A method of constructing iterated forcing notions that has a scope of applications similar to Shelah's oracle-c.c. is presented. This method yields a consistency result on homomorphisms of quotient algebras of the Boolean algebra $\mathscr{P}(\omega)$. Also, it is shown to be relatively consistent with ZFC that the Boolean algebra of Lebesgue measurable subsets of the unit interval has no projective lifting.

0. INTRODUCTION

To begin with, we consider an example. By Bor we denote the Boolean algebra of Borel subsets of the unit interval $(0, 1)$, by $\mathscr{L}$ the ideal of sets of measure zero. A Boolean homomorphism $\underline{H}$: Bor$/\mathscr{L} \to$ Bor is called a *Borel lifting* of the measure algebra iff $\underline{H}([X]) \in [X]$ for every equivalence class $[X] \in$ Bor$/\mathscr{L}$. It is well known that if CH holds, then there exists a Borel lifting of the measure algebra (see [O]). Shelah showed that it is relatively consistent with ZFC that no such lifting exists (see [Sh1]). His proof can be outlined as follows: Start with a model for $V = L$, iterate c.c.c. forcing notions with finite supports. At stage α, guess a function $\underline{H}_\alpha$ and, by iterating at this stage a forcing notion $\mathbf{P}_\alpha$ that adds a new Borel set X without adding a Borel set Y such that $\underline{H}_\alpha([X])$ could be Y, make sure that $\underline{H}_\alpha$ cannot be extended to a Borel lifting. Using a $\diamondsuit_{\omega_2}$-sequence of potential names for the guessing of the $\underline{H}_\alpha$'s, every potential Borel lifting will have been destroyed somewhere along the way, so in the final model there are no such objects.

But—if $\underline{H}_\alpha$ was destroyed by X, could it not happen that a candidate Y for $\underline{H}_\alpha([X])$ is added at some later stage of the iteration? Of course it could, unless we take some extra care. Shelah devised his oracle-c.c. method just for that purpose. First of all, Shelah observed that at stage α we can make sure that not only does forcing with $\mathbf{P}_\alpha$ not add a candidate Y for $\underline{H}_\alpha([X])$, but also forcing with $\mathbf{P}_\alpha \times \mathbf{Q}$ (where $\mathbf{Q}$ is a countable forcing notion) does not add such a Y. Furthermore, he observed that if $\diamondsuit$ holds in the intermediate model

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V_α, then one can formulate a condition which he calls $\overline{M}$-c.c. such that if the remainder of the iteration satisfies $\overline{M}$-c.c., then no candidate Y for $\underline{H}_\alpha([X])$ is added at a later stage. Forcing notions that satisfy the $\overline{M}$-c.c. resemble the Cohen forcing rather closely. The '$\overline{M}$' in '$\overline{M}$-c.c.' is a parameter that depends on the particular situation. It is called an 'oracle' lending the name to Shelah's method.

Naturally the question arises whether, instead of constructing a suitable oracle $\overline{M}$, we could require at stage $\alpha + 1$ that the remainder of the iteration does not add other than Cohen reals over $V_{\alpha+1}$. A c.c.c. forcing notion that does not add other than Cohen reals will be called *harmless.*

In §2 of this paper a method of constructing harmless forcing notions is presented. In §§3–6 we show how this method yields Theorems A and B stated below.

Our first theorem involves the Boolean algebra $\mathscr{P}(\omega)$ of all subsets of the set ω of natural numbers. All ideals in $\mathscr{P}(\omega)$ considered in this paper are assumed to be proper and to contain the ideal Fin of finite subsets of ω. We often identify elements of $\mathscr{P}(\omega)$ (resp. $\mathscr{P}(B)$, where B is some infinite subset of ω) with their characteristic functions. So the product topology on 2^ω (resp. 2^B) induces a topology on $\mathscr{P}(\omega)$ (resp. $\mathscr{P}(B)$). All concepts like 'Borel sets,' 'continuous function' etc. mentioned in connection with Theorem A refer to this topology.

Let $\mathscr{J} \subset \mathscr{P}(\omega)$ be an ideal. A function $F: \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ is said to *preserve intersections* $\bmod \mathscr{J}$, iff for every $A, B \in \mathscr{P}(\omega)$:

(i) $A\Delta B \in \text{Fin} \Rightarrow F(B)\Delta F(B) \in \mathscr{J}$, and

(ii) $F(A \cap B)\Delta(F(A) \cap F(B)) \in \mathscr{J}$.

Let $B \subset \omega$ be an infinite set. A function $F: \mathscr{P}(B) \to \mathscr{P}(\omega)$ is said to be $\mathscr{J}$-*trivial* (or *trivial*, if the choice of $\mathscr{J}$ follows from the context), if there exists a continuous function $F^*: \mathscr{P}(B) \to \mathscr{P}(\omega)$ such that $F^*(A)\Delta F(A) \in \mathscr{J}$ for all $A \subset B$. We call F $\mathscr{J}$-*semitrivial,* if B is the union of finitely many pairwise disjoint sets $B_0, \dots, B_{k-1}$ such that $F \restriction \mathscr{P}(B_i)$ is $\mathscr{J}$-trivial for every $i < k$.

0.1 **Definition.** By AT we abbreviate the following statement: "For every ideal $\mathscr{J} \subset \mathscr{P}(\omega)$ of class Σ^1_1, for every function $F: \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ which preserves intersections $\bmod \mathscr{J}$, and for every uncountable family $\mathscr{B} \subset \mathscr{P}(\omega)$ of pairwise almost disjoint infinite subsets of ω there exists a $B \in \mathscr{B}$ such that $F \restriction \mathscr{P}(B)$ is $\mathscr{J}$-semitrivial."

Theorem A. *Let V be a model for a sufficiently large fragment of* ZFC, *and let κ be a regular uncountable cardinal in V so that $V \models$ 'GCH $+\ \kappa$ is not the successor of a cardinal of cofinality $\omega + \Diamond_\kappa(\{\alpha < \kappa : \text{cf}(\alpha) = \omega_1\})$.' Then there exists a c.c.c. forcing notion* **R** *of cardinality κ in V such that $V^{\mathbf{R}} \models$ '$2^\omega = \kappa$& AT.'*

The statement AT looks a bit awkward; let us familiarize with it by introducing some of its close relatives and descendants. The most distinguished ancestor of AT is the following statement which I abbreviate by TA: "Every automorphism of $\mathscr{P}(\omega)/\text{Fin}$ is trivial." Shelah calls an automorphism of $\mathscr{P}(\omega)/\text{Fin}$ trivial, iff it is induced by a bijection $\sigma: \omega - a \to \omega - b$, where a and b are finite sets (see [Sh2]). In fact, the proof of Theorem A follows very closely Shelah's proof that TA is relatively consistent with ZFC (see [Sh2]). Also, it is there that Shelah expounded his oracle-c.c. method). By analyzing Shelah's proof, it is not hard to see that the method presented here allows us to show the relative consistency of TA with $2^\omega > \omega_2$. However, I do not know the answer to the following.

0.2 *Question.* Does AT imply TA?

A partial answer to the above question was obtained by B. Veličković [V1]. He showed that the following is relatively consistent with ZFC: "MA + there exists a nontrivial automorphism of $P(\omega)/\text{Fin}$ + for every automorphism $\underline{H}$ of $P(\omega)/\text{Fin}$ the set $\{A: \underline{H} \restriction P(A)/\text{Fin} \text{ is trivial}\}$ is nonempty."

Shelah's concept of triviality is very closely related to ours. As a matter of fact, a substantial part of [Sh2] is devoted to the proof of the following lemma—although the lemma itself is not explicitly stated in Shelah's paper.

0.3 **Lemma.** *An automorphism $\underline{F}$ of $\mathscr{P}(\omega)/\text{Fin}$ is induced by a bijection $\sigma: \omega - a \to \omega - b$ (where a, b are finite) iff there exists a continuous function $F: \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ such that the following diagram commutes*

$$\begin{array}{ccc} F: \mathscr{P}(\omega) & \longrightarrow & \mathscr{P}(\omega) \\ \big\downarrow \pi & & \big\downarrow \pi \\ \underline{F}: \mathscr{P}(\omega)/\text{Fin} & \longrightarrow & \mathscr{P}(\omega)/\text{Fin}, \end{array}$$

where π is the natural projection of $\mathscr{P}(\omega)$ onto $\mathscr{P}(\omega)/\text{Fin}$.

It is not hard to see that F as in 0.3 preserves intersections mod Fin.

Inspired by Shelah's result and Veličković's paper [V], I formulated a statement which I used to abbreviate CSP, and proved in my Ph.D. Thesis [J1] its relative consistency with ZFC.

Let $\mathscr{J}, \mathscr{M} \subset \mathscr{P}(\omega)$, where $\mathscr{J}$ is an ideal. By $\text{CSP}(\mathscr{M}, J)$ we abbreviate the statement: "For every function $F: \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ which preserves intersections mod $\mathscr{J}$ there exists a $B \in \mathscr{M}$ such that the function $F \restriction \mathscr{P}(B)$ is $\mathscr{J}$-trivial."

CSP is the statement: "For every comeagre subfamily $\mathscr{M}$ of $\mathscr{P}(\omega)$ and every ideal $\mathscr{J} \in \Sigma^1_1$ the statement $\text{CSP}(\mathscr{M}, J)$ holds."

I do not know the answer to the following.

0.4 *Question.* Do any of the following implications hold: AT $\to$ CSP or CSP $\to$ AT?

At least the following is easy:

0.5 *Claim.* AT implies that $\text{CSP}(\mathscr{P}(\omega) - \text{Fin}, \mathscr{J})$ holds for every ideal $\mathscr{J} \in \Sigma^1_1$. $\square$

Now let us mention some of the consequences of AT. By ω^* we denote the remainder in the Čech-Stone compactification of the countable discrete space ω. Under CH, the space $\omega^* \times \omega^*$ is a continuous image of the space ω^*, and the problem whether this can be shown in ZFC was open for many years. In [J2] it was shown that $\text{CSP}(\mathscr{P}(\omega) - \text{Fin}, \text{Fin})$ implies that for no natural number n the space $(\omega^*)^{n+1}$ is a continuous image of the space $(\omega^*)^n$. By 0.5, the same follows from AT.

A closed subset X of ω^* is called a *P-set*, if for every countable family $\mathscr{U}$ of open supersets of X there is an open superset V of X so that $V \subset \bigcap \mathscr{U}$. E. K. van Douwen and J. van Mill asked (see [vM, p. 537]) whether one can show in ZFC the existence of a nowhere dense P-subset X of ω^* so that X is homeomorphic to ω^*. Again, CH implies the existence of such a set. It was shown in [J3] and published in [J4] that AT (resp. its consequence AKF defined under 0.13) contradicts the existence of a set X as above.

Denote by

$$I_1 = \left\{ A \subset \omega \colon \limsup_{n \to \infty} \frac{|A \cap n|}{n} = 0 \right\}$$

the ideal of sets of density zero, and by

$$I_{\log} = \left\{ A \subset \omega \colon \limsup_{n \to \infty} \frac{\sum_{m \in A \cap n} \frac{1}{m+1}}{\log n} = 0 \right\}$$

the ideal of sets of logarithmic density zero. Erdös and Ulam asked whether CH implies that the algebras $\mathscr{P}(\omega)/I_1$ and $\mathscr{P}(\omega)/I_{\log}$ are not isomorphic (see [E] for a detailed history of the problem). It was shown in [JK] that under CH the algebras $\mathscr{P}(\omega)/I_1$ and $\mathscr{P}(\omega)/I_{\log}$ are isomorphic. On the other hand, as shown in [J1] and published in [J5], AT implies that $\mathscr{P}(\omega)/I_1$ is not isomorphic to $\mathscr{P}(\omega)/I_{\log}$.

If CH holds, then every Boolean algebra of cardinality $\leq 2^\omega$ is embeddable into $\mathscr{P}(\omega)/\mathrm{Fin}$. The situation changes dramatically under AT.

0.6 **Theorem.** *Suppose* AT *holds and let* $\mathscr{I}$ *be an ideal. Suppose furthermore that either there is a measurable cardinal and* $\mathscr{I}$ *is a Borel set, or there is a supercompact cardinal and* $\mathscr{I}$ *is a projective set. Then* $\mathscr{P}(\omega)/\mathscr{I}$ *can be embedded into* $\mathscr{P}(\omega)/\mathrm{Fin}$ *iff* $\mathscr{I}$ *is generated by* Fin *and at most one co-infinite set.*

Notice that by Theorem A, if the existence of a measurable resp. supercompact cardinal is consistent, then it is consistent with AT. I want to give the proof of 0.6 here, since it is a fairly typical example of how to work with AT.

Proof of 0.6. We shall not need the full force of AT here, but only its consequence ATF, which is the following statement: "For every homomorphism of Boolean algebras $\underline{H}\colon P(\omega)/\mathrm{Fin} \to P(\omega)/\mathrm{Fin}$ and every uncountable family $\mathscr{B}$ of pairwise almost disjoint infinite subsets of ω there exist a $B \in \mathscr{B}$ and a continuous function H so that the following diagram commutes:

$$\begin{array}{ccc} H\colon P(B) & \longrightarrow & P(\omega) \\ \downarrow \pi & & \downarrow \pi \\ \underline{H} \restriction P(B)/\mathrm{Fin}\colon P(B)/\mathrm{Fin} & \longrightarrow & P(\omega)/\mathrm{Fin} \end{array}$$

where π is the canonical projection of $P(\omega)$ onto $P(\omega)/\mathrm{Fin}$."

A function H as above will be called a continuous *lifting* of $\underline{H} \restriction P(B)/\mathrm{Fin}$.

0.7 **Proposition.** AT *implies* ATF.

Proof. First observe that Fin is an analytic subset of $P(\omega)$. Moreover, notice that if $\underline{H}$ is any function that maps $P(\omega)/\mathrm{Fin}$ into itself, then $\underline{H}$ has some (not necesarily continuous) lifting H_1. Also, if $\underline{H}$ is a homomorphism, then H_1 preserves intersections mod Fin. Now apply AT for $\mathcal{J} = \text{Fin}$, $F = H_1$, and $\mathcal{B}$ to get $B \in \mathcal{B}$ and H^* so that the restriction of H^* to $P(B_i)$ is continuous for all B_i in a certain finite partition of B into $B_0, \dots, B_{k-1}$. Now define for $A \subset B$:

$$H(A) = \bigcup_{i=0}^{k-1} H^*(A \cap B_i).$$

Since the functions $\cap: P(\omega) \times P(\omega) \to P(\omega)$ and $\bigcup_{i=0}^{k-1}: (P(\omega))^k \to P(\omega)$ are continuous, H is a composition of continuous functions. Also, since $\underline{H}$ is a homomorphism, H is a lifting of $\underline{H} \restriction P(B)/\text{Fin}$. □

We say that an ideal $\mathcal{J}$ is *trivial below* a subset $B \subset \omega$, iff there exists an $A \subset B$ so that $\mathcal{J} \cap P(B)$ is generated by $(\text{Fin} \cap P(B)) \cup \{A\}$. In particular, Fin is trivial below every B, and if $B \in \mathcal{J}$, then $\mathcal{J}$ is trivial below B. We denote

$$\text{Tr}(\mathcal{J}) = \{B \subset \omega : \mathcal{J} \text{ is trivial below } B\}.$$

The following lemma is crucial for our results.

0.8 **Lemma.** *Suppose $\underline{H}: P(\omega)/\text{Fin} \to P(\omega)/\text{Fin}$ is a homomorphism and $H: P(B) \to P(\omega)$ is a continuous lifting of the restriction of $\underline{H}$ to $P(B)/\text{Fin}$. Then $\pi_{\text{Fin}}^{-1}\text{Ker}(\underline{H})$ is trivial below B.*

The proof of 0.8 is already implicit in [Sh2] and [V]. We sketch it for the convenience of the reader.

Proof. Let $\underline{H}$, H be as in the lemma. To simplify the notation, we prove the lemma for the special case $B = \omega$.

Since $P(\omega)$ is a compact metric space, every continuous function which maps $P(\omega)$ into itself is uniformly continuous. Hence there exist sequences of natural numbers $\langle n_k : k \in \omega \rangle$ and $\langle m_k : k \in \omega \rangle$, and a sequence $\langle H_k : k \in \omega \rangle$ of functions so that for all k:

(i) $H_k: P(n_k) \to P(m_k)$,

(ii) If $x \subset n_{k+1}$, then $H_{k+1}(x) \cap m_k = H_k(x \cap n_k)$, and

(iii) $\forall X \subset \omega\ H(X) = \bigcup_{k \in \omega} H_k(X \cap n_k)$.

We call a sequence $\langle H_k : k \in \omega \rangle$ that satisfies (i)–(iii) an *approximation* of H. From now on we fix n_k, m_k and H_k as above.

0.9 **Definition.** Let $k^* > k$. A subset $c \subset [n_k, n_{k^*})$ is called a $[k, k^*)$-*stabilizer*, if $H(x \cup c \cup X) \Delta H(y \cup c \cup X) \subset k^*$ for every $X \subset \omega - n_{k^*}$ and $x, y \subset n_k$.

0.10 *Claim.* For every $k \in \omega$ there exists a $[k, k^*)$-stabilizer $c \subset [n_k, n_{k^*})$ for some $k^* > k$.

Proof. This easily follows from the fact that $H(X)\Delta H(Y) \in \text{Fin}$ whenever $X \Delta Y \in \text{Fin}$. For a more detailed proof see [J2, Proposition 4]. □

Now use the claim to find an increasing sequence of indices $\langle p(k) : k \in \omega \rangle$ and a sequence $\langle c_k : k \in \omega \rangle$ so that $c_k \subset [n_{p(k)}, n_{p(k+1)})$ is a $[p(k), p(k+1))$-stabilizer for every $k \in \omega$. Also, let

$A_0 = \bigcup_{k\in\omega}[n_{p(2k)}, n_{p(2k+1)})$,
$A_1 = \bigcup_{k\in\omega}[n_{p(2k+1)}, n_{p(2k+2)})$.
Define for $x \subset [n_{p(k)}, n_{p(k+1)})$:

$$H_{k^*}(x) = H_{p(k+2)}(c_{k-1} \cup x \cup c_{k+1}) \cap [m_{p(k)}, m_{p(k+2)}).$$

Since H is a lifting of a homomorphism, it is not hard to see that for $X \subset A_0$ the set $H(X)$ differs from $\bigcup_{k\in\omega} H_{2k^*}(x \cap [n_{p(2k)}, n_{p(2k+1)})) \cap H(A_0)$ only by a finite set. An analogous representation can be found for the restriction of H to $P(A_1)$. Since H preserves unions mod finite sets, almost all of the functions H_{k^*} preserve unions. Now it is not hard to see that the ideal $\pi_{\mathrm{Fin}}^{-1}\mathrm{Ker}(\underline{H})$ is generated by Fin and the set $\bigcup_{k\in\omega}\{x \in [n_{p(k)}, n_{p(k+1)}): H_{k^*}(\{x\}) = \varnothing\}$. □

0.11 **Lemma.** *Suppose* ATF *holds, and* $\mathscr{I}$ *is an ideal so that* $\mathrm{Tr}(\mathscr{I})$ *has the Baire property. Then the algebra* $P(\omega)/\mathscr{I}$ *can be isomorphically embedded into* $P(\omega)/\mathrm{Fin}$ *iff* $\mathscr{I}$ *is trivial below* ω.

Proof. We need the following result of S. A. Jalali-Naini and M. Talagrand.

0.12 *Claim.* Suppose $\mathscr{J}$ is an ideal that has the Baire property. Then there exists a sequence $\langle u_n : n \in \omega\rangle$ of pairwise disjoint finite subsets of ω so that no union of infinitely many sets u_n is in $\mathscr{J}$.

Proof. See [T]. □

Now notice that for every ideal $\mathscr{I}$ the family $\mathrm{Tr}(\mathscr{I})$ is either an ideal or is equal to $P(\omega)$. In the latter case, $\mathscr{I}$ is trivial below ω, i.e. $\mathscr{I}$ is generated by $\mathrm{Fin}\cup\{A\}$, where A is some coinfinite subset of ω (possibly finite). Let σ be the enumeration of $\omega - A$ in increasing order. Then the function $H: P(\omega) \to P(\omega)$ defined by $H(X) = \sigma^{-1}X$ for $X \subset \omega$ is a lifting of an isomorphism that maps $P(\omega)/\mathscr{I}$ onto $P(\omega)/\mathrm{Fin}$. This proves the "if" direction.

If $\mathscr{I}$ is nontrivial below ω, then $\mathrm{Tr}(\mathscr{I})$ is a proper ideal. In this case, let $\langle u_n : n \in \omega\rangle$ be a sequence as in Claim 0.12 for $\mathscr{J} = \mathrm{Tr}(\mathscr{I})$. Fix an uncountable family $\mathscr{A}$ of pairwise almost disjoint infinite subsets of ω. For $A \in \mathscr{A}$ define $B(A) = \bigcup\{u_n : n \in A\}$. Then $\mathscr{B} = \{B(A) : A \in \mathscr{A}\}$ is an uncountable family of pairwise almost disjoint subsets of ω such that $B \notin \mathrm{Tr}(\mathscr{I})$ for every $B \in \mathscr{B}$. Suppose $P(\omega)/\mathscr{I}$ can be isomorphically embedded into $P(\omega)/\mathrm{Fin}$. Then there exists a homomorphism $\underline{H}: P(\omega)/\mathrm{Fin} \to P(\omega)/\mathrm{Fin}$ such that $\pi_{\mathrm{Fin}}^{-1}\mathrm{Ker}(\underline{H}) = \mathscr{I}$. By ATF, there exists a $B \in \mathscr{B}$ so that the restriction of $\underline{H}$ to $P(B)/\mathrm{Fin}$ has a continuous lifting. By Lemma 0.8, the ideal $\mathscr{I}$ is trivial below B. We reached a contradiction that concludes the proof of 0.11. □

Now observe that if the predicate '$X \in I$' can be expressed by a formula of class Σ^1_n, then the predicate '$Y \in \mathrm{Tr}(Y)$', which means

$$\exists Z\ Z \in \mathscr{I} \& \forall X \subset Y\ X \in \mathscr{I} \leftrightarrow X - Z \in \mathrm{Fin}$$

can be expressed by a Σ^1_{n+2}-formula. Hence $\mathrm{Tr}(\mathscr{I})$ is a projective set whenever $\mathscr{I}$ is projective, and is of class Σ^1_2 whenever $\mathscr{I}$ is Borel. If there is a measurable cardinal, then every set of class Σ^1_2 has the Baire property. It follows from work of Martin, Steel, and Woodin (see [MSW]), which in turn is based on work of Foreman, Magidor, and Shelah, that the existence of a supercompact cardinal implies that all projective sets have the Baire property. This together with 0.11 proves Theorem 0.6. □

To conclude our discussion of consequences of Theorem A, let us consider the following statement AKF which was used in [J4].

0.13 **Definition.** By AKF we abbreviate the following statement: "For every homomorphism $\underset{\sim}{H}: \mathscr{P}(\omega)/\mathrm{Fin} \to \mathscr{P}(\omega)/\mathrm{Fin}$ and every uncountable family $\mathscr{B}$ of pairwise almost disjoint subsets of ω there exists a $B \in \mathscr{B}$ so that

$$B \in \mathrm{Tr}(\{A \subseteq \omega: A/\mathrm{Fin} \in \mathrm{Ker}(\underset{\sim}{H})\})."$$

It follows immediately from 0.8 that ATF implies AKF.

0.14 *Question.* Does AKF imply ATF?

At the end of this section we return to the example we started with, i.e., to liftings of the measure algebra. By $\mathscr{B}^1_\infty$ we denote the family of all projective subsets of $(0, 1)$.

Theorem B. *Let V be a model for a sufficiently large fragment of* ZFC, *and let κ be a regular cardinal in V so that*

$$V \models \text{'GCH} + \kappa \textit{ is not the successor of a cardinal of cofinality } \omega + \diamondsuit_\kappa(\{\alpha < \kappa: \mathrm{cf}(\alpha) = \omega_1\}).\text{'}$$

Let $\mathscr{I}$ be either the ideal of meagre subsets of $(0, 1)$, or the ideal of subsets of $(0, 1)$ of Lebesgue measure zero. Then there exists a c.c.c. forcing notion $\mathbf{R}$ of cardinality κ in V such that:

$$V^{\mathbf{R}} \models \text{'}\textit{there is no Boolean homomorphism } H: \mathrm{Bor} \to \mathscr{B}^1_\infty \textit{ such that } \mathrm{Ker}(H) = \mathscr{I} \textit{ and } H(X)\Delta X \in \mathscr{I} \textit{ for every } X \in \mathrm{Bor}.\text{'}$$

0.15 *Remark.* It is likely that stronger results than Theorem B can be obtained by methods discussed here. In particular, $\mathscr{B}^1_\infty$ is not a σ-algebra (it contains much bigger σ-algebras than Bor though). I am convinced that Theorem B remains true if we replace $\mathscr{B}^1_\infty$ by the σ-subalgebra of $\mathscr{P}((0, 1))$ generated by the projective sets. I did not strive for maximum generality, but intended to provide a transparent demonstration of what can be done with the forcing technique introduced in this paper. The reader will notice that our proof of Theorem B goes beyond the possibilities of oracle-c.c., even if we are content with iterations of length ω_2.

0.16 *Remark.* If one adds just ω_2 Cohen reals side by side to a model of CH, then in the resulting model there are Borel liftings, and AT fails. The former was shown by T. Carlson [C], the latter by S. Shelah and J. Steprāns [ShSt1]. This shows that the iterations constructed in this paper are *not* equivalent to direct products of enough Cohen forcings. Surprisingly, it is still open whether AT or the absence of Borel liftings follows from $2^\omega > \omega_2$. The methods of [C] and [ShSt1] do not work if *more* than ω_2 Cohen reals are added.

0.17 *Remark.* There are alternative ways to prove the consistency of statements related to AT. S. Shelah and J. Steprāns [ShSt2] showed that the Proper Forcing Axiom (PFA) implies TA, and A. Krawczyk [Kr] independently extracted from [J1] a proof of the relative consistency of a weak version of AT with Martin's Axiom (MA) + $2^\omega = \omega_2$. Recently Boban Veličković showed in [V1] that

OCA + MA implies TA (where OCA stands for the Open Coloring Axiom as defined in [To, p. 72]). Then he and I observed that ATF and the consequences of AT mentioned before 0.6 also follow from OCA + MA by similar arguments [J6]. This work is still in progress, and at present it is not known whether the full AT or the nonexistence of Borel liftings are consistent with MA or even consequences of MA + OCA.

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1. Terminology and some basic facts

It is expected that the reader is well versed in the technique of iterated forcing. Most of our teminology is fairly standard, but some of our conventions are not so widespread, and a few are idiosyncratic. These we shall discuss in the present section.

In this paper, we shall only consider iterations with finite support. The αth iterand will always be denoted by $\mathbf{P}_\alpha$, its name by $\dot{\mathbf{P}}_\alpha$, and the iteration of the first α stages by $\mathbf{R}_\alpha$. When we are going to construct a specific iterand—like in §4—we allow objects of various nature to form the underlying set. However, when discussing iterations we always assume that the underlying set of $\mathbf{P}_\alpha$ is an ordinal. Since all supports are finite, we are entitled to work with *determined conditions*. Thus an element r of $\mathbf{R}_\kappa$ will be a finite partial function whose domain is contained in κ and such that $r(\xi)$ is an ordinal in the underlying set of $\mathbf{P}_\xi$ for every $\xi \in \mathrm{dom}(r)$. The partial order on $\mathbf{P}_\beta$ (which is usually not an element of the ground model) is denoted by $<_\beta$. In particular, the elements of a two-stage iteration $\mathbf{P} * \dot{\mathbf{P}}_1$ are pairs of ordinals, and $\langle \xi, \xi_1 \rangle \le \langle \eta, \eta_1 \rangle$ iff $\xi \le_0 \eta$ and $\xi \Vdash$'$\xi_1 \le_1 \eta_1$.' In this way, we shall know the underlying set of $\mathbf{R}_\kappa$ right from the outset. Recall that what we construct iteratively is the partial order, not necessarily the underlying set. We shall write $\mathrm{supp}(r)$ instead of $\mathrm{dom}(r)$ to denote the support of a condition r. The intermediate model $V^{\mathbf{R}_\alpha}$ will be denoted by V_α.

By *Cohen forcing* we mean any nontrivial countable forcing notion. Cohen forcing is denoted by $\mathbf{Q}$ throughout this paper. If B is an infinite subset of ω, then a *Cohen real relative to* B is a subset C of B such that the set

$\mathbf{G} = \{s : s$ is a finite partial function from B into $\{0, 1\}$ and $s(i) = 1$ iff $i \in \mathrm{dom}(s) \& i \in C\}$ is a generic subset of the forcing notion,

$\{s : s$ is a finite partial function from B into $\{0, 1\}\}$ partially ordered by reverse inclusion. A *Cohen real* is a Cohen real relative to ω.

We write $\mathbf{P} \subseteq \mathbf{P}_1$, if the underlying set of $\mathbf{P}$ is contained in the underlying set of $\mathbf{P}_1$ and the partial order of $\mathbf{P}_1$ extends the partial order of $\mathbf{P}$.

We shall consider also a stronger relation. Define $\mathbf{P} \ll \mathbf{P}_1$ iff $\mathbf{P} \subseteq \mathbf{P}_1$ and $p \perp_{\mathbf{P}} q$ implies $p \perp_{\mathbf{P}_1} q$ for all $p, q \in \mathbf{P}$ (i.e., no two elements incompatible in $\mathbf{P}$ become compatible in $\mathbf{P}_1$).

If $\mathscr{D}$ is a class, then we write $\mathbf{P} \ll_{\mathscr{D}} \mathbf{P}_1$ iff $\mathbf{P} \ll \mathbf{P}_1$ and for all $D \in \mathscr{D}$, if D is a predense subset of $\mathbf{P}$, then D remains predense in $\mathbf{P}_1$. In particular, $\mathbf{P} \ll_V \mathbf{P}_1$ means that $\mathbf{P}$ is completely embedded into $\mathbf{P}_1$.

1.1 **Lemma.** *Let D be any class. The relation '$\ll_{\mathscr{D}}$' has the following properties:*

(a) *'$\ll_{\mathscr{D}}$' is transitive.*

(b) *If $\langle \mathbf{P}_\xi : \xi < \kappa \rangle$ is a sequence such that $\mathbf{P}_\xi \ll_{\mathscr{D}} \mathbf{P}_\eta$ for all $\xi < \eta$, then for every $\xi < \kappa$ there holds: $\mathbf{P}_\xi \ll_{\mathscr{D}} \bigcup_{\xi<\kappa} \mathbf{P}_\xi$.*

(c) *If $\mathbf{R} = \mathbf{P} * \dot{\mathbf{P}}_1$, then $\mathbf{P} \ll_V \mathbf{R}$.*

(d) *If $\mathbf{R}_\kappa$ is an iteration of forcing notions $\langle \dot{\mathbf{P}}_\alpha : \alpha < \kappa \rangle$, and $\lambda < \kappa$, then $\mathbf{R}_\lambda \ll_V \mathbf{R}_\kappa$, where $\mathbf{R}_\lambda = \{r \restriction \lambda : r \in \mathbf{R}_\kappa\}$.*

(e) *If $\mathbf{R} \ll_V \mathbf{R}_1$, then there exists an $\mathbf{R}$-name $\dot{\mathbf{P}}$ such that $\mathbf{R} * \dot{\mathbf{P}}$ and $\mathbf{R}_1$ are equivalent forcing notions.*

(f) *If ϕ is a Σ_0-formula or a Σ^1_2-formula whose parameters are $\mathbf{P}$-names for reals, and if $\mathbf{P} \ll_V \mathbf{P}_1$, then for every $p \in P$: $p \Vdash_{\mathbf{P}} \phi$ iff $p \Vdash_{\mathbf{P}_1} \phi$.*

(g) *Suppose $\mathbf{R}^- \ll_V \mathbf{R}$, and $\dot{\mathbf{P}}^-$ is an $\mathbf{R}^-$-name for a forcing notion with underlying set in V. If $\dot{\mathbf{P}}$ is an $\mathbf{R}$-name such that $\Vdash_{\mathbf{R}}$ '$\dot{\mathbf{P}}^- \ll_{V^{\mathbf{R}^-}} \dot{\mathbf{P}}$', then $\mathbf{R}^- * \dot{\mathbf{P}}^- \ll_V \mathbf{R} * \dot{\mathbf{P}}$.*

(h) *If $\mathbf{R}^- \ll_V \mathbf{R}$ and $\mathbf{P}$ is an $\mathbf{R}^-$-name for a forcing with underlying set from V, then $\mathbf{R}^- * \dot{\mathbf{P}} \ll_V \mathbf{R} * \dot{\mathbf{P}}$.*

(i) *If $\mathbf{R}^- \ll_V \mathbf{R}$ and $\mathbf{P}$ is any forcing notion, then $\mathbf{R}^- \times \mathbf{P} \ll_V \mathbf{R} \times \mathbf{P}$.*

Before we prove the above lemma, let us fix one more bit of terminology. If $\mathbf{R} \ll_V \mathbf{R}_1$, then we shall write $\dot{\mathbf{R}}_1/\mathbf{R}$ for an $\mathbf{R}$-name for a forcing notion such that $\mathbf{R}_1$ and $\mathbf{R} * (\dot{\mathbf{R}}_1/\mathbf{R})$ are equivalent. The forcing $\mathbf{R}_1/\mathbf{R}$ is called the *remainder of* $\mathbf{R}_1$ *over* $\mathbf{R}$. It is determined up to equivalence of forcing notions. In particular, if $\mathbf{R}_\kappa$ and $\mathbf{R}_\lambda$ are as in point (d) of the lemma, then we write $\dot{\mathbf{R}}_{\lambda\kappa}$ instead of $\mathbf{R}_\kappa \dot{/} \mathbf{R}_\lambda$.

Proof of 1.1. Points (a)–(d) are obvious, (e) is a classical result (see e.g. [G, p. 457, Theorem 1]). Point (f) is a consequence of (e) and of absoluteness of formulas of the respective classes. Point (h) is a special case of (g), but it will be convenient to prove it first. Since $\dot{\mathbf{P}}$ is an $\mathbf{R}^-$-name, no condition in $\mathbf{R}/\mathbf{R}^-$ is mentioned in the construction of $\mathbf{P}$. Therefore, $\mathbf{R} * \dot{\mathbf{P}}$ is equivalent to the forcing $\mathbf{R}^- * (\mathbf{R}\dot{/}\mathbf{R}^- \times \dot{\mathbf{P}})$. Now it is not hard to see that (h) follows from the product lemma. Notice that in our approach a product $\mathbf{R} \times \mathbf{P}$ is simply an instance of a two-step iteration $\mathbf{R} * \dot{\mathbf{P}}$. By the same token, (i) is an instance of (h).

It remains to prove (g). Let $\mathbf{R}^-$, $\mathbf{R}$, $\dot{\mathbf{P}}^-$, $\dot{\mathbf{P}}$ be as in the assumptions. The relation $\Vdash_{\mathbf{R}}$ '$\dot{\mathbf{P}}^- \ll \dot{\mathbf{P}}$' implies that $\mathbf{R} * \dot{\mathbf{P}}^- \ll \mathbf{R} * \dot{\mathbf{P}}$. From (a) and (h) (for $\mathscr{D} = \varnothing$) we infer that $\mathbf{R}^- * \dot{\mathbf{P}}^- \ll \mathbf{R} * \dot{\mathbf{P}}$. Now let $\langle r, p \rangle \in \mathbf{R} * \dot{\mathbf{P}}$, and suppose D is a predense subset of $\mathbf{R}^- * \dot{\mathbf{P}}^-$. It follows from (h) that D remains predense in $\mathbf{R} * \dot{\mathbf{P}}^-$. Hence

$r \Vdash_{\mathbf{R}}$ 'The set $D^* = \{p_1 : \exists r_1 \langle r_1, p_1 \rangle \in D \,\&\, r_1 \in \dot{\mathbf{G}}^-\}$ is predense in $\dot{\mathbf{P}}^-$,'

where $\dot{\mathbf{G}}^-$ denotes the canonical $\mathbf{R}^-$-name for the generic subset of $\mathbf{R}^-$. Observe that D^* has an $\mathbf{R}^-$-name. Since $r \Vdash_{\mathbf{R}}$ '$\dot{\mathbf{P}}^- \ll_{V^{\mathbf{R}^-}} \dot{\mathbf{P}}$,' we have in particular

$r \Vdash_{\mathbf{R}}$ '$\dot{D}^*$ is a predense subset of $\dot{\mathbf{P}}$.'

It follows that there are conditions $r_2 \le r$ and $\langle r_1, p_1 \rangle \in D$ such that $r_2 \not\perp_{\mathbf{R}} r_1$ and $r_2 \Vdash_{\mathbf{R}}$ '$p_1 \not\perp_{\mathbf{P}} p$.' This means that $\langle r, p \rangle \not\perp_{\mathbf{R} * \dot{\mathbf{P}}} \langle r_1, p_1 \rangle$. We have thus shown that the set D remains predense in $\mathbf{R} * \dot{\mathbf{P}}$. □

Now we turn our attention to some topological facts about $\mathscr{P}(C)$, where C is an infinite subset of ω. The canonical basis of $\mathscr{P}(C)$ consists of the sets of the form U_s, where s is a finite function from some subset of C into $\{0, 1\}$. The open set U_s contains all those $A \subset C$ whose characteristic function extends s. A subset $K \subset \mathscr{P}(C)$ has the *Baire property* iff there is an open subset $U(K)$ of $\mathscr{P}(C)$ so that the symmetric difference $U(K)\Delta K$ is the union of countably many nowhere dense subsets of $\mathscr{P}(C)$. We say that K is of *first Baire category* (or *meagre*), if K has the Baire property and $U(K) = \varnothing$. If K has the Baire property and $U(K) \neq \varnothing$, then K is said to be of *second Baire category.* If K is of second Baire category and $U(K)$ is dense open, then K is called *comeagre.* A function $F: \mathscr{P}(C) \to \mathscr{P}(\omega)$ is said to be *Baire measurable,* if every preimage of an open set has the Baire property. If $A, B \subset \omega$, and $\mathscr{J} \subset \mathscr{P}(\omega)$ is an ideal, then we shall frequently write $A =_{\mathscr{J}} B$ instead of $A\Delta B \in \mathscr{J}$. Two functions $F, F^*: \mathscr{P}(C) \to \mathscr{P}(\omega)$ are said to be $\mathscr{J}$*-equivalent* on a set $K \subset \mathscr{P}(C)$ iff $F(A) =_{\mathscr{J}} F^*(A)$ for all $A \in K$. The following lemma will be needed in §4.

1.2 **Lemma.** *Suppose $F, F^*: \mathscr{P}(C) \to \mathscr{P}(\omega)$ are $\mathscr{J}$-equivalent on a comeagre subset K_0 of $\mathscr{P}(C)$. If F preserves intersections* mod $\mathscr{J}$*, and F^* is Baire measurable, then F is $\mathscr{J}$-semitrivial.*

Proof. Let $M = \bigcup\{U(F^{*-1}(U_s))\Delta F^{*-1}(U_s): s \in 2^{<\omega}\}$. The set M is meagre, and if we put $K_1 = \mathscr{P}(C) - M$, then it is not hard to see that $F^* \restriction K_1$ is continuous. Therefore, F is $\mathscr{J}$-equivalent to a continuous function on the comeagre set $K_0 \cap K_1$.

1.3 *Claim.* Let K be a comeagre subset of $\mathscr{P}(C)$. There exist subsets A, A_1, B, B_1 of C so that A, B are infinite and disjoint, $A_1 \subset A$, $B_1 \subset B$, and for every $X \subset C - B$ and every $Y \subset C - A$ we have $X \cup B_1 \in K$ and $A_1 \cup Y \in K$.

Proof. Let $\langle M_n: n \in \omega\rangle$ be an increasing sequence of nowhere dense subsets of $\mathscr{P}(C)$ so that $\bigcup_{n\in\omega} M_n \supset \mathscr{P}(C) - K$. We define inductively a sequence $\langle s_n: n \in \omega\rangle$ of finite partial functions from C into $\{0, 1\}$ of pairwise disjoint nonempty domains so that $U_{s_n} \cap M_n = \varnothing$ for every n. The construction goes as follows. Suppose we have already constructed s_m for $m < n$. Let k be such that $\bigcup\{\text{dom}(s_m): m < n\} \subset k$. Let $t_0, \ldots, t_j$ be an enumeration of all functions from $C \cap k$ into $\{0, 1\}$. For $i \leq j$, we choose $s_n^i \neq \varnothing$ so that $\text{dom}(s_n^i) \cap k = \varnothing$,

$s_n^i \supseteq s_n^{i-1}$ (if applicable), and $U_{s_n^i \cup t_i} \cap M_n = \varnothing$. Let $s_n = s_n^j$, and let

$A = \bigcup\{\text{dom}(s_{2n}): n \in \omega\}$,

$B = \bigcup\{\text{dom}(s_{2n+1}): n \in \omega\}$,

$B_1 = \bigcup\{s_{2n}^{-1}\{1\}: n \in \omega\}$,

$A_1 = \bigcup\{s_{2n+1}^{-1}\{1\}: n \in \omega\}$.

Since $M_n \subseteq M_{n+1}$ for all n, this works. □

Now apply the claim for $K = K_0 \cap K_1$. Let A, A_1, B, B_1 be as in the claim. Since F preserves intersections mod $\mathscr{J}$, we have for $X \subset A$: $F(X) =_{\mathscr{J}} F^*(X \cup B_1) \cap F(A)$. Similarly, for $Y \subset C - A$ we have: $F(Y) =_{\mathscr{J}} F^*(A_1 \cup Y) \cap F(C - A)$. Since the functions $X \mapsto X \cup B_1$ and $Z \mapsto Z \cap F(A)$ are continuous, also the function $X \mapsto F^*(X \cup B_1) \cap F(A)$ is continuous on $\mathscr{P}(A)$. Therefore $F \restriction \mathscr{P}(A)$ is trivial. An analogous reasoning shows that also the restriction of F to $\mathscr{P}(C - A)$ is trivial, so we conclude that F is semitrivial. □

At the end of this section I want to mention briefly a few conventions that are quite common in the set-theoretic literature, though not ubiquitous. OR denotes the class of ordinals, Lim the class of limit ordinals. A function $f\colon \mathrm{cf}(\alpha) \to \alpha$ is said to be *normal* in α, if it is nondecreasing, continuous in the order topology, and if its range is cofinal in α. By $H(\omega_1)$ we denote the family of hereditarily countable sets. If M and N are models for some theory, then we write $M \prec_{\Sigma_{100}} N$, iff M is a submodel of N such that if ϕ is a formula of class Σ_{100} with parameters in M, then $M \models \phi$ iff $N \models \phi$. By 'c.u.b.' we abbreviate 'closed unbounded'. If p is an ordered pair, then $(p)_0$ is its first, $(p)_1$ its second element (hence $p = \langle (p)_0, (p)_1 \rangle$). The symbol '$A \dot\cup B = C$' means '$A \cup B = C$ and $A \cap B = \varnothing$.'

We shall frequently speak about reals as codes for Borel or projective sets. It is usually not important how the coding is precisely done, the reader is encouraged to think of his or her own favoured way of coding. However, we require that if the real A codes a Σ^1_n-set, then the formula 'X is an element of the Σ^1_n-set coded by A' is a formula of class Σ^1_n.

Finally, I admit that I frequently blur the distinction between a sequence $\langle x_\alpha\colon \alpha < \kappa \rangle$ and its range $\{x_\alpha\colon \alpha < \kappa\}$ just for the sake of notational convenience. I hope this will not cause confusion.

2. A construction of harmless forcing notions

2.1 **Definition.** A notion of forcing $\mathbf{P}$ is called *harmless*, if $\mathbf{P}$ satisfies the c.c.c. and for every countable $\mathbf{P}_0 \subseteq \mathbf{P}$ there exists a countable $\mathbf{P}_1$ such that $\mathbf{P}_0 \subseteq \mathbf{P}_1 \ll_V \mathbf{P}$.

2.2 **Fact.** If $\mathbf{P}$ is harmless, then every real added by forcing with $\mathbf{P}$ is constructible from a Cohen real over V. $\square$

2.3 **Definition.** Let $\mathbf{R}_\kappa$ be a finite support iteration of forcing notions $\langle \dot{\mathbf{P}}_\alpha\colon \alpha < \kappa \rangle$, and let $S = \{\alpha < \kappa\colon \mathrm{cf}(\alpha) = \omega_1\}$. We say that $\mathbf{R}_\kappa$ is *innocuous*, if the following conditions hold:

(a) The underlying set of $\mathbf{P}_\alpha$ is ω for every $\alpha \in \kappa - S$, and ω_1 for every $\alpha \in S$.

(b) For $\alpha \in S$, there exist in the intermediate model V_α:

- a normal function $f_\alpha\colon \omega_1 \to \alpha$,
- a closed unbounded subset $C_\alpha \subset \omega_1$,
- a sequence of reals $\langle Z^\alpha_\xi\colon \xi < \omega_1 \rangle = Z^\alpha$ such that:

(b1) For every ξ, the set Z^α_ξ is a Cohen real over the intermediate model $V_{f_\alpha(\xi)}$.

(b2) For every $\delta \in C_\alpha$, we have $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{\mathbf{P}}_\alpha \cap \delta \in V_{f_\alpha(\delta)}$.'

(b3) Suppose there is a sequence $\langle V^+_\xi\colon \xi < \omega_1 \rangle$, possibly in some richer world than V_α, such that for all $\xi < \eta < \omega_1$:

- $V_{f_\alpha(\xi)} \subseteq V^+_\xi \subset V^+_\eta$,
- V^+_ξ is a model of a sufficiently large fragment of ZFC,
- Z^α_η is a Cohen real over V^+_η.

Then for every $\delta \in C_\alpha$, we have $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{\mathbf{P}}_\alpha \cap \delta \ll_{V^+_\delta} \dot{\mathbf{P}}_\alpha$.'

Obviously, we cannot quantify in V_α over all sequences $\langle V^+_\xi \rangle$ that may exist in some "richer world." One should therefore interpret (b3) as a condition that holds in V_α and is absolute in a certain sense.

2.4 *Remark.* If we replace in Definition 2.3 "there exists a normal function" by "for all normal functions" we obtain an equivalent definition. Moreover, if $\mathbf{R}_\alpha$ satisfies the c.c.c., then we can always find a normal set C_α in the ground model so that (b2) holds. □

The main result of this section is the following.

2.5 **Theorem.** *Let κ be any cardinal. If $\mathbf{R}_\kappa$ is innocuous, then $\mathbf{R}_\kappa$ is harmless.*

We split the proof of 2.5 into two lemmas.

2.6 **Lemma.** *Let κ be a cardinal and suppose $\mathbf{R}_\kappa$ is innocuous. Then $\mathbf{R}_\kappa$ satisfies the c.c.c.*

Proof. Let $\mathbf{R}_\kappa$ be as in Definition 2.3. It suffices to show inductively that if $\alpha < \kappa$ and $\mathbf{R}_\alpha$ satisfies the c.c.c. then $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{\mathbf{P}}_\alpha$ satisfies the c.c.c.'

For $\alpha \notin S$ this is obvious, since by definition the underlying set of $\mathbf{P}_\alpha$ is countable. So suppose that $\alpha \in S$ and $\mathbf{R}_\alpha$ satisfies the c.c.c. Let f_α and C_α be as in the hypothesis of 2.3(b).

By 2.4 and the inductive hypothesis we may assume that both f_α and C_α are in the ground model.

Let $\dot{D}$ be an $\mathbf{R}_\alpha$-name for a subset of ω_1—the underlying set of $\mathbf{P}_\alpha$—such that $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{D}$ is a maximal antichain in $\dot{\mathbf{P}}_\alpha$.'

There exists a c.u.b. subset $C^1 \subset \omega_1$ such that $C^1 \in V$ and $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{D} \cap \delta \in V_{f_\alpha(\delta)}$' for all $\delta \in C^1$. In V_α we define a function $g \colon \omega_1 \to \omega_1$ as follows: $g(\xi) = \min\{\eta \in D \colon \xi \not\perp_{\mathbf{P}_\alpha} \eta\}$. Since D is maximal, g is well defined. There exists a c.u.b. subset $C \subset \omega_1$ such that $\forall \delta \in C \;\; \forall \xi < \delta \;\; g(\xi) < \delta$. Since $\mathbf{R}_\alpha$ satisfies the c.c.c., we may assume C is in the ground model. For $\delta \in C \cap C^1 \cap C_\alpha$ there holds:

$\Vdash_{\mathbf{R}_\alpha}$ '$\dot{D} \cap \delta$ is a maximal antichain in $\dot{\mathbf{P}}_\alpha \cap \delta \& \dot{D} \cap \delta \in V_{f_\alpha(\delta)}$.'

Applying (b3) with $V_\xi^+ = V_{f_\alpha(\xi)}$ we infer

$\Vdash_{\mathbf{R}_\alpha}$ '$\dot{D} \cap \delta$ is predense in $\dot{\mathbf{P}}_\alpha$.'

It follows that $D = D \cap \delta$. Since $\delta < \omega_1$, we have shown that D is countable. □

Now let κ and $\mathbf{R} = \mathbf{R}_\kappa$ be as above and fix:

- a family $\mathscr{F} = \{f_\alpha \colon \alpha \in S\}$ of functions with common domain ω_1 such that f_α is normal in α,
- a family $\mathscr{C} = \{C_\alpha \colon \alpha \in S\}$ of c.u.b. subsets of ω_1,
- a family of names $\mathscr{Z} = \{\dot{Z}_\xi^\alpha \colon \xi < \omega_1, \alpha \in S\}$ such that $f_\alpha, C_\alpha, \langle Z_\xi^\alpha \colon \xi < \omega_1\rangle$ witness condition 2.3(b). Since $\mathbf{R}$ satisfies the c.c.c., we can assume $\mathscr{F}, \mathscr{C} \subset V$.

Let V^- be a model for a sufficiently large fragment of ZFC such that $V^- \prec_{\Sigma_{100}} V$, and $\mathbf{R}, \mathscr{F}, \mathscr{C}, \mathscr{Z} \in V^-$. Denote $\mathbf{R}^- = V^- \cap \mathbf{R}$.

2.7 **Lemma.** $\mathbf{R}^- \ll_V \mathbf{R}$.

Theorem 2.5 is an immediate consequence of 2.6 and 2.7: apply 2.7 for a countable structure V^-.

Proof of 2.7. First observe that since $V^- \prec_{\Sigma_{100}} V$ we have $\mathbf{R}^- \ll \mathbf{R}$. It remains to show that predense subsets of $\mathbf{R}^-$ remain predense in $\mathbf{R}$. For $\alpha \leq \kappa$ denote $\mathbf{R}_\alpha^- = \mathbf{R}_\alpha \cap V^-$. We show inductively that $\mathbf{R}_\alpha^- \ll_V \mathbf{R}_\alpha$. First notice that if α

is a limit ordinal and $\mathbf{R}_\beta^- \ll_V \mathbf{R}_\beta$ for all $\beta < \alpha$, then $\mathbf{R}_\alpha^- \ll_V \mathbf{R}_\alpha$. So assume $\alpha = \beta + 1$, and $\mathbf{R}_\beta^- \ll_V \mathbf{R}_\beta$. Let $D \subset \mathbf{R}_\alpha^-$ be a predense subset and let $q \in \mathbf{R}_\alpha$. In order to show that there is some $p \in D$ such that $p \not\perp_{\mathbf{R}_\alpha} q$, we distinguish three cases.

First case. $\beta \notin V^-$. The set $D_\beta = \{p \restriction \beta : p \in D\}$ is predense in $\mathbf{R}_\beta^-$, and by the inductive assumption it remains predense in $\mathbf{R}_\beta$. Hence, there exists a $p \in D$ such that $p \restriction \beta \not\perp_{\mathbf{R}_\beta} q \restriction \beta$. Now it suffices to notice that $\mathrm{supp}(p) \subset V^-$ for every $p \in \mathbf{R}_\alpha^-$; hence, $p \restriction \beta = p$ for all $p \in D$.

Second case. $\beta \in V^- \& \beta \notin S$. In this case the underlying set of $\dot{\mathbf{P}}_\beta$ is ω; hence, if all names are nice names we have $\dot{\mathbf{P}}_\beta \cap V^- = \dot{\mathbf{P}}_\beta$ and $\mathbf{R}_\beta^- * \dot{\mathbf{P}}_\beta = \mathbf{R}_\alpha^-$. We conclude that $\mathbf{R}_\alpha^- \ll_V \mathbf{R}_\alpha$.

Third case. $\beta \in V^- \& \beta \in S$. In this case, $V^- \cap \omega_1 = \xi_0$ for some ordinal ξ_0. If $\xi_0 = \omega_1$, then we may reason as in the second case. Let us assume that ξ_0 is a countable ordinal. In a sense, $\dot{\mathbf{P}}_\beta \cap V^- = \dot{\mathbf{P}}_\beta \cap \xi_0$.

2.8 *Claim.* $\Vdash_{\mathbf{R}_\beta}$ '$\dot{\mathbf{P}}_\beta \cap \xi_0 \ll_{V^{\mathbf{R}_\beta^-}} \dot{\mathbf{P}}_\beta$.'

By Lemma 1.1(g), the proof in the third case is an immediate consequence of 2.8.

Proof of 2.8. Since $\mathscr{F}, \mathscr{C}, \beta \in V^-$, we have also $f_\beta, C_\beta \in V^-$. Hence, $V^- \Vdash$ 'C_β is a c.u.b. subset of ω_1 and f_β is a normal function from ω_1 into β.' It follows that the range of $f_\beta \restriction \xi_0$ is cofinal in $\gamma = \sup(V^- \cap \beta)$ and that the set $C_\beta \cap \xi_0$ is cofinal in ξ_0. Hence, $\xi_0 \in C_\beta$ and $f_\beta(\xi_0) = \gamma$. Applying 2.3(b) for $V_\xi^+ = V_{f_\beta(\xi)}$, we infer that $\Vdash_{\mathbf{R}_\beta}$ '$\dot{\mathbf{P}}_\beta \cap \xi_0 \ll_{V_\gamma} \dot{\mathbf{P}}_\beta$,' so all the more $\Vdash_{\mathbf{R}_\beta} \dot{\mathbf{P}}_\beta \cap \xi_0 \ll_{V^{\mathbf{R}_\beta^-}} \dot{\mathbf{P}}_\beta$. This concludes the proof of 2.8, and hence, of 2.7 and 2.5. □

Unlike such notions as chain conditions and properness, the property of being an innocuous iteration is in general not inherited by the remainder $\mathbf{R}_{\alpha\kappa}$.

It may even happen that for an innocuous forcing $\mathbf{R}_\kappa$ and some $\alpha \in S$ we have

$\Vdash_{\mathbf{R}_\alpha}$ '$\dot{\mathbf{R}}_{\alpha\kappa}$ is not harmless.' But still the following holds.

2.9 **Lemma.** *Let* $\mathbf{R}_\kappa$ *be innocuous, and let* $\alpha \in \kappa - S$. *Then* $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{\mathbf{R}}_{\alpha\kappa}$ *is equivalent to an innocuous iteration.*'

Lemma 2.9 is a special case of a more general fact. To formulate it we need some notation. Let $V^- \subset V$. We say that V^- is *ω-ended*, if for every ordinal $\alpha \in V - V^-$ either $\mathrm{cf}(\alpha) = \omega$ or $\sup(V^- \cap \alpha) < \alpha$.

2.10 *Claim.* For every ordinal α there exists an ω-ended structure $V^- \prec_{\Sigma_{100}} V$ such that $|V^-| = \omega_1$, and $\omega_1 \cup \{\alpha\} \subset V^-$.

Proof. Observe that if W is a structure of cardinality ω_1, then the set $E(W) = \{\alpha \in \mathrm{OR} : \alpha \notin W \& \sup(W \cap \alpha) = \alpha\}$ is of cardinality at most ω_1. Thus, V^- may be constructed as the union of a Σ_{100}-elementary chain $(W_n)_{n\in\omega}$ of submodels of V such that $E(W_n) \subset W_{n+1}$ for all $n \in \omega$. □

2.11 **Lemma.** *Suppose* $\mathbf{R}_\kappa$ *is an innocuous iteration of forcing notions* $\langle \mathbf{P}_\alpha : \alpha < \kappa \rangle$, *and* $\alpha \in S \cap \kappa$. *Let* $V^- \prec_{\Sigma_{100}} V$ *be an ω-ended structure of cardinality* ω_1 *so that* $\omega_1 \subset V^-$ *and* $\alpha, \mathbf{R}_\kappa \in V^-$.

Then $\mathbf{R}_{\alpha+1} \cap V^- \lessdot_V \mathbf{R}_\kappa$ *and* $\Vdash_{\mathbf{R}_{\alpha+1}\cap V^-}$ '$\mathbf{R}_\kappa/(\mathbf{R}_{\alpha+1} \cap V^-)$ *is equivalent to an innocuous iteration.*'

Proof. Like on previous occasions, denote $\mathbf{R}^- = V^- \cap \mathbf{R}^-_{\alpha+1}$. The first part of the lemma is true by Lemma 2.7. Notice that if $\beta \in V^-$, then $\dot{\mathbf{P}}_\beta \in V^-$ and $\dot{\mathbf{P}}_\beta \subset V^-$ (since $\dot{\mathbf{P}}_\beta \in H(\omega_2)$). Also, notice that by elementarity, in V^- there are sequences $\mathscr{F}$, $\mathscr{C}$, $\mathscr{Z}$ witnessing that $\mathbf{R}_\kappa$ is innocuous.

Let $\pi: \alpha+1 \cap V^- \to \gamma$ be an order isomorphism. Notice that if $\beta \in \mathrm{dom}(\pi)$ and $\mathrm{cf}(\beta) \le \omega_1$, then $\mathrm{cf}(\pi(\beta)) = \mathrm{cf}(\beta)$. If $\beta \in \mathrm{dom}(\pi)$ and $\mathrm{cf}(\beta) > \omega_1$, then $\mathrm{cf}(\pi(\beta)) = \omega$, since V^- is ω-ended.

Also, let $\sigma: \alpha+1 - V^- \to \gamma'$ be an order isomorphism. It is not hard to see that $\mathrm{cf}(\beta) \ge \mathrm{cf}(\sigma(\beta))$ for all β, and if $\mathrm{cf}(\beta) \ge \omega_1$, then $\mathrm{cf}(\beta) = \mathrm{cf}(\sigma(\beta))$.

Finally, we define a function $\tau: \kappa \to \kappa$ as follows:

$$\tau(\beta) = \begin{cases} \gamma + \gamma' + \gamma'' & \text{for } \beta = \alpha + 1 + \gamma'', \\ \pi(\beta) & \text{for } \beta \in V^- \cap \alpha + 1, \\ \gamma + \sigma(\beta) & \text{for } \beta \in \alpha + 1 - V^-. \end{cases}$$

Define a function $\overline{\tau}: \mathbf{P}_\kappa \to \mathbf{P}_\kappa$ as follows:

If $p \in \mathbf{P}_\kappa$, then $\mathrm{dom}(\overline{\tau}(p)) = \tau[\mathrm{dom}(p)]$, and if $\beta \in \mathrm{dom}(p)$ is such that $p(\beta) = \xi$, then $\overline{\tau}(p)(\tau(\beta)) = \xi$.

If $\xi, \eta \in \mathbf{P}_\beta$, and $p \Vdash_{\mathbf{R}_\beta}$ '$\xi <_\beta \eta$,' then $\xi, \eta \in \mathbf{P}_{\tau(\beta)}$, and $\overline{\tau}(p) \Vdash_{\mathbf{R}_{\tau(\beta)}}$ '$\xi <_{\overline{\tau}(\beta)} \eta$.' In other words, we change the order of iteration. We are entitled to do so, because if $\beta \in V^-$, then every ordinal mentioned in defining the order $<_\beta$ of $\mathbf{P}_\beta$ also is an element of V^- (the name for $<_\beta$ is an object of size ω_1). It is not hard to see that $\overline{\tau}$ is an automorphism of $\mathbf{P}_\kappa$.

Now our task reduces to showing that $\Vdash_{\overline{\tau}[\mathbf{R}^-_{\alpha+1}]}$ '$\overline{\tau}[\mathbf{R}_\kappa]/\overline{\tau}[\mathbf{R}^-_{\alpha+1}]$ is equivalent to an innocuous iteration.' Clearly, $\Vdash_{\overline{\tau}[\mathbf{R}^-_{\alpha+1}]}$ '$\overline{\tau}[\mathbf{R}_\kappa]/\overline{\tau}[\mathbf{R}^-_{\alpha+1}] = \overline{\tau}[\dot{\mathbf{R}}_{\alpha+1,\kappa}]$.'

So we reduced 2.11 to 2.9, except that we must first convince ourselves that $\overline{\tau}[\mathbf{R}_\kappa]$ is still an innocuous iteration.

2.12 *Claim.* $\overline{\tau}[\mathbf{R}_\kappa]$ is an innocuous iteration.

Proof. Since τ preserves cofinality ω_1, the image $\overline{\tau}[\mathbf{R}_\kappa]$ satisfies 2.3(a). Now suppose $\beta \in \kappa \cap S$, and let μ be such that $\tau(\mu) = \beta$. Let f_μ, C_μ, $\langle Z^\mu_\xi : \xi < \omega_1\rangle$ be the witnesses of 2.3(b) for the iteration $\mathbf{R}_\kappa$ at stage μ.

We distinguish two cases:

First case. $\mu \notin V^- \cap \alpha + 1$. There exists a $\delta < \mu$ such that $\{\eta : \delta < \eta < \mu\} \cap V^- \cap \alpha + 1 = \varnothing$, hence, $f_\mu(\xi) \notin V^- \cap \alpha + 1$ for sufficiently large ξ.

By 2.4, we may assume that $f_\mu(\xi) \notin V^- \cap \alpha + 1$ for all ξ. We concentrate on the case where $\mu \le \alpha$. The case $\mu > \alpha$ is even simpler.

Now $\tau(f_\mu(\xi)) = \gamma + \sigma(f_\mu(\xi))$ and $\lim_{\xi<\omega_1} \tau(f_\mu(\xi)) = \beta$. We know that

$$\Vdash_{\mathbf{R}_\mu} \text{`}\dot{\mathbf{P}}_\mu \cap \delta \in V_{f_\mu(\delta)} \quad \text{for every } \delta \in C_\mu.\text{'}$$

All the more, since $\overline{V}_{\tau(f_\mu(\delta))} \supset V_{f_\mu(\delta)}$, we have

$$\Vdash_{\overline{\tau}[\mathbf{R}_\mu]} \text{`}\dot{\mathbf{P}}_\beta \cap \delta \in \overline{V}_{\tau(f_\mu(\delta))}\text{'} \quad \text{for } \delta \in C_\mu.$$

Here, $\overline{\tau}[\mathbf{R}_\mu]$ is the iteration of the first β steps in the new order, and $\overline{V}_{\tau(f_\mu(\delta))}$ is an intermediate model according to the new order of iteration. We shall write $\overline{\mathbf{R}}_\beta$ instead of $\overline{\tau}[\mathbf{R}_\mu]$ in the sequel.

Now consider 2.3(b3) for $\mathbf{R}_\mu$. Notice that the sequence $\langle V_\xi^+ = \overline{V}_{\tau(f_\mu(\xi))} : \xi < \omega_1 \rangle$ satisfies the hypothesis of 2.3(b3), since Z_ξ^μ is still a Cohen real over V_ξ^+ for every $\xi < \omega_1$! Indeed, by assumption, Z_ξ^μ is a Cohen real over $\mathbf{R}_\xi$, where $\xi = \sup(V^- \cap f_\mu(\xi))$; so Z_ξ^μ must have been added at some stage between ξ and μ.

Now in the names $\dot{\mathbf{P}}_\eta$ for $\eta \in V^- - \xi$ none of the objects constructed at stages between ξ and μ is mentioned (recall that $V^- \cap \mu \subset \xi$), so Z_ξ^μ meets all dense sets of the Cohen forcing whose names mention only conditions in $\mathbf{R}_{f_\mu(\xi)}$ and in $\mathbf{R}^-_{\alpha+1}$, in other words, conditions in $\overline{\mathbf{R}}_{\tau(f_\mu(\xi))}$.

It follows that if we put: $\overline{f}_\beta = \tau \circ f_\mu$, $\overline{C}_\beta = C_\mu$, and $\overline{Z}_\xi^\beta = Z_\xi^\mu$ for all ξ, then conditions (b1) and (b2) are satisfied. The new condition (b3) is now in fact a weakening of the corresponding condition at the old stage μ.

Second case. $\mu \in V^- \cap \alpha+1$. We show that $\tau \circ f_\mu$, C_μ and $\langle Z_\xi^\mu : \xi < \omega_1 \rangle$ witness 2.3(b). Clearly, these are objects of the right kind, and since $\overline{V}_{\tau(f_\mu(\xi))} \subset V_{f_\mu(\xi)}$, condition (b1) holds. Since all forcing conditions mentioned in the name $\dot{\mathbf{P}}_\mu$ are in V^-, also (b2) holds. It is somewhat less obvious that (b3) goes through; after all, there might be more "richer worlds" which contain $\overline{V}_{\tau(f_\mu(\xi))}$ than there are "richer worlds" that contain $V_{f_\mu(\xi)}$. Once more we should remember how we arrive at (b3). In V_μ the forcing $\mathbf{P}_\mu$ has a certain property ϕ which may be expressed by a formula of ZFC. This formula, although a bit technical, certainly does not involve more than 100 alternative quantifiers.

By Σ_{100}-elementarity, $\overline{V}_\beta \models \phi(\dot{\overline{\mathbf{P}}}_\beta)$. ($\phi$ has more parameters of course. These are omitted here.) Now by a metamathematical reasoning—which is still expressible in ZFC, as long as we settle for models for a fixed fragment of ZFC—we deduce the actual wording of (b3) as an absoluteness property of ϕ. This concludes the proof of the claim. □

So we are left with the

Proof of Lemma 2.9. Condition 2.3(a) follows from the fact that $\omega_1^V = \omega_1^{V_\alpha}$.

Let $\beta \in \kappa - (\alpha + 1)$ be an ordinal of cofinality ω_1, and let f_β, C_β and $\langle Z_\xi^\beta : \xi < \omega_1 \rangle$ be witnesses for (b).

Let $\zeta = \min\{\xi : f_\beta(\xi) \geq \alpha\}$. We define $\tilde{f}_\beta(\xi) = f_\beta(\zeta + \xi) - \alpha$. Clearly, $\tilde{f}_\beta$ is a normal function, and $\tilde{f}_\beta(\xi) = f_\beta(\xi)$ for cofinally many ξ. It is not hard to see that $\tilde{f}_\beta$, $\langle Z_{\zeta+\xi}^\beta : \xi < \omega_1 \rangle$ and a suitable final segment of C_β witness condition 2.3(b). □

3. Reduction of Theorem A to Main Lemma A

3.1 Main Lemma A. *Suppose* CH *holds and we are given*:

- *an ideal* $\mathscr{J}$ *which is a* Σ_1^1*-subset of* $\mathscr{P}(\omega)$;
- *a family* $\{B_\xi : \xi < \omega_1\}$ *of pairwise almost disjoint infinite subsets of* ω;
- *a function* $F : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ *that preserves intersections* mod $\mathscr{J}$ *such that for every* $\xi < \omega_1$ *the restriction* $F \restriction \mathscr{P}(B_\xi)$ *is not semitrivial*;
- *a sequence* $\langle V(\xi) : \xi < \omega_1 \rangle$ *of transitive models for a sufficiently large fragment of* ZFC *such that*:
  . $V(\xi) \subset V(\eta)$ *for* $\xi < \eta < \omega_1$,

. $H(\omega_1) = \bigcup_{\xi<\omega_1} V(\xi) \cap H(\omega_1)$,
. *whenever* $D \subset \omega_1$, *then the set* $ND = \{\xi < \omega_1 : D \cap \xi \in V_\xi\}$ *contains a c.u.b. subset of* ω_1;
- *a sequence* $\langle Z_\xi : \xi < \omega_1 \rangle$ *of subsets of* ω *such that* Z_ξ *is a Cohen real over* $V(\xi)$ *for* $\xi < \omega_1$.

Then there exist
- *a normal function* $g: \omega_1 \to \omega_1$;
- *sequences* $\langle A_\xi : \xi < \omega_1 \rangle$ *and* $\langle C_\xi : \xi < \omega_1 \rangle$ *of pairwise almost disjoint subsets of* ω;
- *a sequence* $\langle \mathbf{P}^\xi : \xi < \omega_1 \rangle$ *of countable forcing notions*;
- *a* $\mathbf{P} = \bigcup_{\xi<\omega_1} \mathbf{P}^\xi$*-name* $\dot{X}$ *such that*

(T1) $$\mathscr{J} \in V(g(0));$$

(T2) $$\mathbf{P}^{\xi+1}, A_\xi, C_\xi, F(A_\xi), F(C_\xi) \in V(g(\xi+1)) \quad \text{for } \xi < \omega_1;$$

(T3) $$\mathbf{P}^\xi \ll_{V(g(\xi))} \mathbf{P}^\eta \quad \text{for } \xi < \eta < \omega_1;$$

(T4) $$|\mathbf{P}| = \omega_1;$$

(T5) $$\mathbf{P}^\lambda = \bigcup_{\xi<\lambda} \mathbf{P}^\xi \quad \text{for } \lambda \in \mathrm{Lim} \cap \omega_1;$$

(T6) *Denote by* $\mathbf{Q}$ *the Cohen forcing. Then*

$\Vdash_{\mathbf{P}}$ '$\forall \xi < \omega_1 \dot{X} \cap C_\xi =_{\mathrm{Fin}} A_\xi$,' *and for every* $\mathbf{P} \times \mathbf{Q}$*-name* $\dot{Y}$ *for a subset of* ω *there exists* $\xi < \omega_1$ *such that for all* η *between* ξ *and* ω_1:
$\Vdash_{\mathbf{P}^{\eta+1} \times \mathbf{Q}}$ '$\dot{Y} \cap F(C_\eta) \neq_{\mathscr{J}} F(A_\eta)$.'

Moreover, these objects may be constructed in V *in such a way that the following holds: Suppose there is a richer world* V^+ *which knows of an increasing sequence* $\langle V(\xi)^+ : \xi < \omega_1 \rangle$ *of models for a sufficiently large fragment of* ZFC *such that for all* $\xi < \omega_1$:
- $V(\xi) \subseteq V(\xi)^+$,
- Z_ξ *is a Cohen real over* $V(\xi)^+$.

Then the following holds in V^+:

(T3+) $$\mathbf{P}^\xi \ll_{V(g(\xi))^+} \mathbf{P}^\eta \quad \text{for } \xi < \eta < \omega_1.$$

3.2 *Claim.* Suppose $\mathbf{P}$ is as in 3.1. Then $\mathbf{P}$ satisfies the c.c.c.

Proof. Like Lemma 2.6. It is exclusively for the sake of this proof that we require ND to contain a c.u.b. whenever $D \subset \omega_1$. □

The remainder of this section is devoted to the demonstration how Theorem A follows from the Main Lemma A. We start with a universe V that satisfies the hypothesis of Theorem A (e.g. L will do), and fix $\kappa \in V$ as in Theorem A. We want to construct an innocuous iteration $\mathbf{R}_\kappa$ of forcing notions $\langle \dot{\mathbf{P}}_\alpha : \alpha < \kappa \rangle$. Since the underlying set of $\mathbf{R}_\kappa$ is determined by Definition 2.3, we know beforehand the objects which may become $\mathbf{R}_\kappa$-names. These objects will be called *potential* $\mathbf{R}_\kappa$*-names* (see §4 for a detailed discussion of potential names for reals. There, a different forcing notion is considered, but the idea remains the same). Since $\mathbf{R}_\kappa$ will satisfy the c.c.c., we may fix the following sequences before the actual construction of $\mathbf{R}_\kappa$:
- a sequence $\langle \dot{F}'_\alpha : \alpha < \kappa \rangle$ of certain potential $\mathbf{R}_\kappa$-names for functions $F: \mathscr{P}(\omega) \to \mathscr{P}(\omega)$;

- a sequence $\langle \dot{\mathscr{J}}_\alpha : \alpha < \kappa \rangle$ of all potential $\mathbf{R}_\kappa$-names for codes of analytic ideals in $\mathscr{P}(\omega)$;

- a sequence $\langle \dot{\mathscr{B}}_\alpha : \alpha < \kappa \rangle$ of all potential $\mathbf{R}_\kappa$-names for families of cardinality ω_1 of pairwise almost disjoint infinite subsets of ω.

Moreover, since $V \Vdash \diamondsuit_\kappa(S)$ (where $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega_1\}$), we can arrange these in such a way that for every triple $\langle \dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}} \rangle$ of potential $\mathbf{R}_\kappa$-names for a function from $\mathscr{P}(\omega)$ into $\mathscr{P}(\omega)$, a code for an analytic ideal and a family of cardinality ω_1 of pairwise almost disjoint subsets of ω respectively, the set $S(\dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}}) = \{\alpha \in S : \dot{F} \restriction \mathscr{P}(\omega) \cap V_\alpha = \dot{F}' \restriction \mathscr{P}(\omega) \cap V_\alpha \,\&\, \dot{\mathscr{J}} = \dot{\mathscr{J}}_\alpha \,\&\, \dot{\mathscr{B}} = \dot{\mathscr{B}}_\alpha\}$ is stationary in κ. By $\dot{F}_\alpha$ we shall denote $\dot{F}'_\alpha \restriction \mathscr{P}(\omega) \cap V_\alpha$.

Now we construct inductively the forcing notions $\dot{\mathbf{P}}_\alpha$. If $\alpha \in \kappa - S$, then $\dot{\mathbf{P}}_\alpha$ is an $\mathbf{R}_\alpha$-name for the Cohen forcing. If $\alpha \in S$, then we check whether the following five conditions hold:

(i) $\dot{F}_\alpha$ is an $\mathbf{R}_\alpha$-name for a function from $\mathscr{P}(\omega)$ into $\mathscr{P}(\omega)$;

(ii) $\dot{\mathscr{J}}_\alpha$ is an $\mathbf{R}_\alpha$-name for a code of an analytic ideal;

(iii) $\dot{\mathscr{B}}_\alpha$ is an $\mathbf{R}_\alpha$-name for a family of cardinality ω_1 of pairwise almost disjoint infinite subsets of ω;

(iv) $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{F}_\alpha$ preserves intersections mod $\dot{\mathscr{J}}_\alpha$';

(v) $\Vdash_{\mathbf{R}_\alpha}$ '$\forall B \in \dot{\mathscr{B}}_\alpha\ \dot{F}_\alpha \restriction B$ is not semitrivial.'

If one of these conditions does *not* hold, then $\dot{\mathbf{P}}_\alpha$ will be a product with finite supports of countable forcing notions.

If (i)–(v) hold, then $\Vdash_{\mathbf{R}_\alpha}$ '$\langle \dot{F}_\alpha, \dot{\mathscr{J}}_\alpha, \dot{\mathscr{B}}_\alpha \rangle$ is a counterexample to AT.'

In this situation we want to design $\dot{\mathbf{P}}_\alpha$ so as to destroy the counterexample. We fix a function $f_\alpha : \omega_1 \to \alpha$ normal in α, and a sequence $\mathscr{Z}^\alpha = \langle Z^\alpha_\xi : \xi < \omega_1 \rangle$ of reals so that in V_α the hypotheses of the Main Lemma A are satisfied with $F_\alpha = F$, $\mathscr{B}_\alpha = \{B_\xi : \xi < \omega_1\}$, $\mathscr{J}_\alpha = \mathscr{J}$, and $V(\xi) = V_{f(\xi)}$ for all ξ. The one hypothesis of 3.1 which may not hold in V_α is the Continuum Hypothesis. If it does hold, then we let $\dot{\mathbf{P}}_\alpha$ be an $\mathbf{R}_\alpha$-name for a forcing notion $\mathbf{P}$ so that (T1)–(T6) and (T3+) are satisfied in V_α. If CH does not hold in V_α (i.e. if $\alpha > \omega_2$), then we choose an ω-ended structure $V^- \prec_{\Sigma_{100}} V$ of cardinality ω_1 such that $\omega_1 \subset V^-$ and everything relevant is an element of V^-. In particular, we want $\dot{\mathscr{Z}}^\alpha$, $\dot{F}_\alpha$, $\dot{\mathscr{J}}_\alpha$, and $\dot{\mathscr{B}}_\alpha$ to be $\mathbf{R}^-_\alpha$-names, where $\mathbf{R}^-_\alpha = \mathbf{R}_\alpha \cap V^-$.

3.3 *Claim.* $\Vdash_{\mathbf{R}^-_\alpha}$ '$\langle \dot{F}_\alpha, \dot{\mathscr{J}}_\alpha, \dot{\mathscr{B}}_\alpha \rangle$ is a counterexample to AT.'

Proof. Notice that e.g. a formula like $\Vdash_{\mathbf{P}}$ '$\forall x \exists y \phi(x, y)$' can be rewritten as: $\forall\mathbf{P}$-name $\dot{x}\exists\mathbf{P}$-name $\dot{y}(\Vdash_{\mathbf{P}}$ '$\phi(\dot{x}, \dot{y})$'). So the claim follows from the fact that $V^- \prec_{\Sigma_{100}} V$. □

It follows that $V^-_\alpha = V^{\mathbf{R}^-_\alpha}$ satisfies the hypothesis of the Main Lemma A. We let $\dot{\mathbf{P}}_\alpha$ be an $\mathbf{R}^-_\alpha$-name for a forcing notion $\mathbf{P}$ so that (T1)–(T5) and (T3+) are satisfied in V^-_α. Since the forcing notions $\mathbf{P}^\xi$ are countable, we may without loss of generality assume that the underlying set of $\mathbf{P}^\xi$ is $\omega \times \xi$ for all ξ.

This finishes the description of $\mathbf{R}_\kappa$. We show that it works.

3.4 *Claim.* $\mathbf{R}_\kappa$ is innocuous.

Proof. Clearly, 2.3(a) is satisfied. Also, if $\alpha \in S$ and at stage α we did not have to deal with a counterexample, then enumerating $\mathbf{P}_\alpha$ in such a way that the product of the first δ Cohen forcing notions is enumerated by δ for all

$\delta \in \mathrm{Lim} \cap \omega_1$, we easily see that $\mathbf{P}_\alpha \cap \delta \ll_{V_\alpha} \mathbf{P}_\alpha$ for all $\delta \in \mathrm{Lim} \cap \omega_1$, i.e., $C_\alpha = \mathrm{Lim} \cap \omega_1$ witnesses 2.3(b). The normal function f_α and the sequence $\mathscr{Z}^\alpha$ are irrelevant in this case.

If $\alpha \in S$ is such that we invoke the Main Lemma A for the construction of $\mathbf{P}_\alpha$, then let f_α and $\mathscr{Z}^\alpha$ be as used in the construction of $\mathbf{P}_\alpha$. Let $C_\alpha = \{\xi : \omega \times \xi = \xi \,\&\, \mathbf{P}^\xi \in V(\xi)\}$. It follows from (T5) that C_α is a c.u.b. subset of ω_1, and since we assumed the underlying set of $\mathbf{P}^\xi$ to be $\omega \times \xi$, it follows that 2.3(b2) holds. Condition 2.3(b1) is true by assumption, and 2.3(b3) corresponds to (T3+). □

3.5 **Corollary.** $\mathbf{R}_\kappa$ *satisfies the c.c.c.* □

3.6 **Corollary.** $V_\kappa \Vdash$ '$2^\omega = \kappa$.' □

The next lemma is the last brick needed for the proof of Theorem A.

3.7 **Lemma.** $V_\kappa \Vdash AT$.

Proof. Suppose not. Then there is a triple of $\mathbf{R}_\kappa$-names $\langle \dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}} \rangle$ such that $\Vdash_{\mathbf{R}_\kappa}$ '$\langle \dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}} \rangle$ is a counterexample to AT.'

Notice that by absoluteness of Σ^1_1-formulas, (i)–(iv) are satisfied at all stages $\alpha \in S(\dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}})$.

3.8 *Claim.* There is an $\alpha \in S(\dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}})$ so that (v) holds.

Proof. Suppose not. Then in V_κ the following holds: $\forall \alpha \in S(\dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}}) \exists B^\alpha \in \mathscr{B} \exists k \in \omega \exists A_0^\alpha, \dots, A_{k-1}^\alpha \exists \tilde{F}_0^\alpha, \dots, \tilde{F}_{k-1}^\alpha B^\alpha = A_0^\alpha \dot{\cup} \cdots \dot{\cup} A_{k-1}^\alpha \& \forall i < k F \restriction \mathscr{P}(A_i^\alpha) =_{\mathscr{J}} \tilde{F}_i^\alpha \& \tilde{F}_i^\alpha$ is continuous.

A continuous function on the separable space $\mathscr{P}(A)$ is uniquely determined by its values on a countable dense subset, say on $\mathrm{Fin} \cap \mathscr{P}(A)$. This allows us to encode continuous functions by reals. Since $\mathrm{cf}(\alpha) = \omega_1$ for $\alpha \in S$, and $\mathbf{R}_\alpha$ satisfies the c.c.c., the following holds in V_κ: $\forall \alpha \in S(\dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}}) \exists \beta(\alpha) < \alpha \exists B^\alpha \in \mathscr{B} \exists k \in \omega \exists A_0^\alpha, \dots, A_{k-1}^\alpha \in V_{\beta(\alpha)} \exists \tilde{F}_0^\alpha, \dots, \tilde{F}_{k-1}^\alpha \in V_{\beta(\alpha)}$ $B^\alpha = A_0^\alpha \dot{\cup} \cdots \dot{\cup} A_{k-1}^\alpha \& \forall i < k F \restriction \mathscr{P}(A_i^\alpha) =_{\mathscr{J}} \tilde{F}_i^\alpha \& \tilde{F}_i^\alpha$ is continuous.

The function $\alpha \mapsto \beta(\alpha)$ is regressive, so by the Pressing Down Lemma there is some stationary set $S_1 \subset S(\dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}})$ and a β such that $\beta(\alpha) = \beta$ for all $\alpha \in S_1$. We may assume that the $B^\alpha = B$ and k are always the same on S_1; and since $V_{\beta(\alpha)} \Vdash 2^\omega < \kappa$, we also may assume that $A_i^\alpha = A_i$ and $\tilde{F}_i^\alpha = F_i$ for all $\alpha \in S_1$. But then $F \restriction \mathscr{P}(B)$ is semitrivial in V_α, a contradiction. □

Now let $\alpha \in S(\dot{F}, \dot{\mathscr{J}}, \dot{\mathscr{B}})$ be such that (v) holds at stage α of the construction, i.e., we iterate at stage α a forcing notion $\dot{\mathbf{P}}_\alpha$ which satisfies (T1)–(T6) and (T3+) of the Main Lemma A. Consider the set $X \subset \omega$ whose existence the Main Lemma A postulates. Let $\dot{X}$ be an $\mathbf{R}^-_{\alpha+1}$-name for X, where $\mathbf{R}^-_{\alpha+1} = \mathbf{R}^-_\alpha * \dot{\mathbf{P}}_\alpha$. We show that there does not exist an $\mathbf{R}_\kappa$-name $\dot{Y}$ for a potential value of $F(X)$.

3.9 *Claim.* Let $\mathbf{Q}$ denote Cohen forcing. $\Vdash_{\mathbf{R}^-_{\alpha+1} \times \mathbf{Q}}$ 'for every $Y \subset \omega$ the function $F_\alpha \cup \langle \dot{X}, Y \rangle$ cannot be extended to a function preserving intersections mod $\dot{\mathscr{J}}$.'

Proof. Notice that in our approach to finite support iterations the forcing notions $\mathbf{R}^-_{\alpha+1} \times \mathbf{Q}$ and $\mathbf{R}^-_\alpha * (\dot{\mathbf{P}}_\alpha \times \mathbf{Q})$ are identical. Instead of $\dot{\mathbf{P}}_\alpha$ we shall write

$\dot{\mathbf{P}}$. Let $\dot{Y}$ be an $\mathbf{R}_\alpha^- * (\mathbf{P} \times \mathbf{Q})$-name for a subset of ω. By (T6), there exist a maximal antichain $\{r_i : i \in \omega\}$ in $\mathbf{R}_\alpha^-$, and a set of ordinals $\{\xi_i : i \in \omega\}$ such that for every $i \in \omega$ and $\eta > \xi_i$: $r_i \Vdash_{\mathbf{R}_\alpha^- * (\mathbf{P}^{\eta+1} \times \mathbf{Q})}$ '$\dot{Y} \cap \dot{F}_\alpha(\dot{C}_\alpha) \neq_{\dot{\mathcal{J}}_\alpha} \dot{F}_\alpha(\dot{A}_\eta)$.'

Thus if $\eta > \xi_\omega = \sup\{\xi_i : i \in \omega\}$, then $\Vdash_{\mathbf{R}_\alpha^- * (\mathbf{P}^{\eta+1} \times \mathbf{Q})} \phi_\eta$, where $\phi_\eta =$ '$\dot{A}_\eta \cup \dot{C}_\eta \subset \omega \,\&\, \dot{Y} \cap \dot{F}_\alpha(\dot{C}_\eta) \Delta \dot{F}_\alpha(\dot{A}_\eta) \notin \dot{\mathcal{J}} \,\&\, \dot{X} \cap \dot{C}_\eta \Delta \dot{A}_\eta \in \mathrm{Fin}$.'

Clearly, the proof of the claim will be complete if we show that also $\Vdash_{\mathbf{R}_\alpha^- * (\dot{\mathbf{P}} \times \mathbf{Q})} \phi_\eta$ for some η below ω_1. So fix η between ξ_ω and ω_1, and let $\zeta < \zeta^+ < \alpha$ be such that $f_\alpha(\eta) = \zeta$ and $f_\alpha(\eta+1) = \zeta^+$. From (T2) we infer that $\dot{\mathbf{P}}^{\eta+1}$, $\dot{A}_\eta$, $\dot{C}_\eta$, $\dot{F}_\alpha(\dot{A}_\eta)$, $\dot{F}_\alpha(\dot{C}_\eta)$ are $\mathbf{R}_{\zeta^+}^-$-names; and without loss of generality we may assume that $\dot{\mathcal{J}}$ is an $\mathbf{R}_{\zeta^+}^-$-name, and that $\dot{X}$, $\dot{Y}$ are both $\mathbf{R}_{\zeta^+}^- * (\dot{\mathbf{P}}^{\eta+1} \times \mathbf{Q})$-names. By Lemma 1.1(h), $\mathbf{R}_{\zeta^+}^- * (\dot{\mathbf{P}}^{\eta+1} \times \mathbf{Q}) \ll_V \mathbf{R}_\alpha^- * (\dot{\mathbf{P}}^{\eta+1} \times \mathbf{Q})$.

Notice that ϕ_η is a Π_1^1-formula, hence by Lemma 1.1(f) we have $\Vdash_{\mathbf{R}_{\zeta^+}^- * (\dot{\mathbf{P}}^{\eta+1} \times \mathbf{Q})} \phi_\eta$. Also, by Lemma 1.1(g) and (T3), $\mathbf{R}_{\zeta^+}^- * (\dot{\mathbf{P}}^{\eta+1} \times \mathbf{Q}) \ll_V \mathbf{R}_\alpha^- * (\dot{\mathbf{P}} \times \mathbf{Q})$, hence $\Vdash_{\mathbf{R}_\alpha^- * (\dot{\mathbf{P}} \times \mathbf{Q})} \phi_\eta$. $\square$

Claim 3.9 may be interpreted as follows: "In $V^{\mathbf{R}_{\alpha+1}^-}$, the function F_α cannot be extended onto X to a function preserving intersections mod $\mathcal{J}$, and this remains true in any model obtained by adding one Cohen real to $V^{\mathbf{R}_{\alpha+1}^-}$." Now recall that by Lemma 2.11 we have: $\Vdash_{\mathbf{R}_{\alpha+1}^-}$ '$\mathbf{R}_\kappa / \dot{\mathbf{R}}_{\alpha+1}$ is harmless.'

By 2.2, this implies that every candidate Y for $F(X)$ would be constructible from a Cohen real over $V^{\mathbf{R}_{\alpha+1}^-}$, contradicting 3.9. Therefore, $V_\kappa \Vdash$ 'F does not preserve intersections mod $\mathcal{J}$,' contradicting our initial assumption. We have thus proved 3.7. $\square$

4. Proof of the Main Lemma A

Throughout this section we assume that CH holds and fix $\mathcal{J}$, F, $\{B_\xi : \xi < \omega_1\}$, $\langle V_\xi : \xi < \omega_1 \rangle$, $\langle Z_\xi : \xi < \omega_1 \rangle$ which satisfy the assumptions of the Main Lemma A. Our aim is to construct a function g and sequences $\langle \mathbf{P}^\xi : \xi < \omega_1 \rangle$, $\langle A_\xi : \xi < \omega_1 \rangle$, $\langle C_\xi : \xi < \omega_1 \rangle$ and a $\mathbf{P} = \bigcup_{\xi < \omega_1} \mathbf{P}^\xi$-name $\dot{X}$ for a real such that (T1)–(T6) hold.

The "Moreover ... "-part of the lemma will be a by-product of our construction. Let $\overline{A} = \{A_i : i < \xi \le \omega_1\}$ and $\overline{D} = \{D_i : i < \xi \le \omega_1\}$ be such that $\overline{A} \cup \overline{D}$ is a family of subsets of ω with pairwise finite intersection (the sets A_i, D_i themselves may be infinite, finite or even empty). We define a notion of forcing $\mathbf{P}(\overline{A}, \overline{D})$ as follows:

$\mathbf{P}(\overline{A}, \overline{D}) = \{p : p$ is a partial function from ω into $\{0, 1\}$ which is a union of finitely many functions of the form $1_{A_i} \Delta$ fin and $0_{D_i} \Delta$ fin$\}$.

Here j_B denotes the function with domain B assuming the value j for all its arguments. We say that a function g is of the form $f \Delta \mathrm{fin}$ if $f \Delta g$ is a finite set, i.e., the symmetric difference of the domains of f and g is finite and there are only finitely many common arguments for f and g such that $f(x) \neq g(x)$.

The forcing conditions are partially ordered by reverse inclusion.

The forcing notions $\mathbf{P}^\xi$ which we are going to construct will be of the form $\mathbf{P}^\xi = \mathbf{P}(\overline{A}^\xi, \overline{D}^\xi)$, where $\overline{A}^\xi = \{A_i : i < \xi\}$ and $\overline{D}^\xi = \{D_i : i < \xi\}$. Clearly, $\mathbf{P}^\xi$

thus defined will be countable for every $\xi < \omega_1$. If moreover $\overline{A}^{\omega_1} = \bigcup_{\xi<\omega_1} A_\xi$ or $\overline{D}^{\omega_1} = \bigcup_{\xi<\omega_1} D_\xi$ contains uncountably many infinite sets, then $\mathbf{P}$ will be uncountable, i.e., (T4) will hold. It is worth noting that $p \not\perp q$ holds iff $p \cup q$ is a function; no matter whether the compatibility relation refers to $\mathbf{P}^\xi$ or $\mathbf{P}^\eta$. Thus, $\mathbf{P}^\xi \ll \mathbf{P}^\eta$ for $\xi < \eta$.

The name $\dot{X}$ will be a name for the subset of ω whose characteristic function is $\bigcup \mathbf{G}$, where $\mathbf{G}$ denotes the generic subset of $\mathbf{P}$. Obviously, we can choose $\dot{X}$ to be a $\mathbf{P}^0$-name.

Actually, we shall choose A_i and D_i so that $A_i \cap D_i = \varnothing$. In this case, the union $1_{A_i} \cup 0_{D_i}$ will be a forcing condition; we denote it by Ch_i. For convenience, we denote $C_i = A_i \cup D_i$ for $i < \omega_1$. Notice that $Ch_i \Vdash$ '$\dot{X} \cap C_i = A_i$' and $\Vdash$ '$\dot{X} \cap C_i =_{\mathrm{Fin}} A_i$,' no matter whether we interpret the forcing relation "$\Vdash$" as "$\Vdash^V_{\mathbf{P}}$", "$\Vdash^{V^+}_{\mathbf{P}}$" or "$\Vdash^V_{\mathbf{P}^\xi}$". At this stage of the proof, we have already schematically constructed our forcing notion $\mathbf{P}$. All what remains to do is to choose appropriate A_i's and D_i's and a function g such that (T1)–(T6) hold. So let us relax for a moment and contemplate what we have done so far. A *potential element* of $\mathbf{P} \times \mathbf{Q}$ is a scheme

$$\tilde{p} = \langle h \cup 1_{A_{i(0)-N}} \cup 1_{A_{i(1)-N}} \cup \cdots \cup 1_{A_{i(k)-N}} \cup 0_{D_{i(k+1)-N}} \cup \cdots \cup 0_{D_{i(r)-N}}, q \rangle,$$

where $k, r, N \in \omega$, h is a finite partial function from ω into $\{0, 1\}$, $q \in \mathbf{Q}$, and $i(0), \ldots, i(k+r)$ are pairwise different countable ordinals.

By p we denote the interpretation of the scheme $\tilde{p}$ obtained by substitution of the actually constructed A_i and D_i.

An interpretation p of $\tilde{p}$ is an element of $\mathbf{P} \times \mathbf{Q}$ iff $(p)_0$ is a function.

Similarly, a potential name for a subset of ω is a set of triples: $\dot{y} = \{\langle n, \tilde{p}^n_k, t^n_k \rangle : k, n \in \omega \,\&\, \tilde{p}^n_k$ is a potential element of $\mathbf{P} \times \mathbf{Q} \,\&\, t^n_k \in \{\text{true}, \text{false}\}\}$. A potential name $\dot{y}$ becomes a name, if all schemes $\tilde{p}^n_k$ become conditions in $\mathbf{P} \times \mathbf{Q}$ and for all $n \in \omega$ the set $M_n = \{p^n_k : k \in \omega\}$ becomes a maximal antichain in $\mathbf{P} \times \mathbf{Q}$. Notice that here we tacitly make use of the fact that $\mathbf{P} \times \mathbf{Q}$ is bound to become a c.c.c. forcing notion, as we know from Claim 3.2. Since CH holds, we may arrange in a sequence $\langle y_\xi : \xi < \omega_1 \rangle$ all potential $\mathbf{P} \times \mathbf{Q}$-names for subsets of ω.

Now we are going to construct inductively the function g and the sequences $\langle A_i : i < \omega_1 \rangle$ and $\langle D_i : i < \omega_1 \rangle$.

The construction of an appropriate function g is easy: At successor stages, take care that g grows sufficiently quickly to ensure $Rg(g)$ becomes cofinal in ω_1. At limit stages, take care that g is continuous.

Now suppose we are at stage ξ of the construction, that we know $A_i < D_i$ for $i < \xi$, and $g(\xi)$. If ξ is a limit ordinal, then it may happen that $\mathbf{P}^\xi \notin V_{g(\xi)}$, so we choose an ordinal $\beta < \omega_1$ such that $g(\xi) \leq \beta$, and $\mathbf{P}^\xi, B_\xi \in V_\beta$.

4.1 **Lemma.** *Whenever $C_\xi \subset B_\xi$ is a Cohen real (relative to B_ξ) over V^+_β, and A_ξ, D_ξ are such that $A_\xi \cup D_\xi = C_\xi$, $A_\xi \cap D_\xi = \varnothing$, then* $\mathbf{P}(\overline{A}^\xi, \overline{D}^\xi) \ll_{V^+_\beta} \mathbf{P}(\overline{A}^\xi \cup \{A_\xi\}, \overline{D}^\xi \cup \{D_\xi\})$.

Of course the above lemma does not quite fit into the frame of our proof, since it requires at least some knowledge about the richer world V^+. For that reason we defer its proof for the time being. However, an appropriate C_ξ is readily available in our actual world V; just take $\mathrm{en}[Z_\beta]$, where en is the function enumerating B_ξ in increasing order. So we can carry out the construction entirely in V, not even suspecting that there might be a richer world, where the final product of our efforts has still some nice properties.

We say that a $\mathbf{P}^\xi$-name $\dot{y}$ for a subset of ω *receives attention* at stage ξ, if $\dot{y} = \dot{y}_\zeta$ for some $\zeta \leq \xi$ and $\dot{y} \in V_{g(\xi)}$. Notice that there are at most countably many names $\dot{y}$ receiving attention at stage ξ.

4.2 **Lemma.** *Let $C \subset \omega$ be an infinite set which is almost disjoint from all sets A_i, D_i for $i < \xi$.*

If $F \restriction \mathscr{P}(C)$ is not semitrivial, then there exists $A_\xi \subset C$ such that if $\mathbf{P}^{\xi+1} = \mathbf{P}(\overrightarrow{A}^\xi \cup \{A_\xi\}, \overrightarrow{D}^\xi \cup \{C - A_\xi\})$, then for every $\dot{y}$ which receives attention at stage ξ there holds: $\Vdash_{\mathbf{P}^{\xi+1}\times\mathbf{Q}}$ '$F(C) \cap \dot{y} \Delta F(A_\xi) \notin \mathscr{J}$.'

Notice that if $C^1 \subset B_\xi$ is constructed from Z_β as described above, then either $C = C^1$ or $C = B_\xi - C^1$ satisfies the hypothesis of 4.2 (since $F \restriction \mathscr{P}(B_\xi)$ is not semitrivial). So by 4.1 and 4.2 we find $A_\xi \subset C$ and $D_\xi = C - A_\xi$ such that $\mathbf{P}^\xi \ll_{V_\beta} \mathbf{P}^{\xi+1}$ and (T6) holds for all $\dot{y}$ which receive attention at stage ξ. Choosing $g(\xi + 1) > \beta$ large enough to ensure that (T2) holds, we finish our construction.

It is easily seen that (T1)–(T5) hold. Condition (T6) follows from the fact, that every $\mathbf{P} \times \mathbf{Q}$-name $\dot{y}$ for a real eventually receives attention.

It remains to prove the lemmas.

Proof of 4.1. Denote for the purpose of this proof: $\mathbf{P} = \mathbf{P}(\overrightarrow{A}^\xi, \overrightarrow{D}^\xi)$ and $\mathbf{P}^+ = \mathbf{P}(\overrightarrow{A}^\xi \cup \{A_\xi\}, \overrightarrow{D}^\xi \cup \{D_\xi\})$, for parameters as in the hypothesis of the lemma. It is clear that $\mathbf{P} \ll \mathbf{P}^+$. Now let T be a predense subset of $\mathbf{P}$, and $\tilde{p} = p_0 \cup 1_{A-N} \cup 0_{D-N}$ be a potential element of $\mathbf{P}^+$ (where $p_0 \in \mathbf{P}$). By $p(A, D)$ we denote the interpretation of $\tilde{p}$ for a given choice of A and D. Note that $p(A, D)$ is an element of $\mathbf{P}^+$ iff it is a function. Let $G(\tilde{p}, T) = \{X \subset B_\xi : \forall Y \subset X \exists r \in T p(Y, X - Y)$ is not a function or $p(Y, X - Y) \not\perp r\}$.

4.3 *Claim.* $G(\tilde{p}, T)$ contains a dense open subset of $\mathscr{P}(B_\xi)$.

Proof. As in §0, we identify subsets of B_ξ with functions $f \in \{0, 1\}^{B_\xi}$. Let s be a function on a finite initial segment of B_ξ into $\{0, 1\}$. We show that there is a function t from a finite initial segment of B_ξ such that $f \in G(\tilde{p}, T)$ whenever $t \subset f$. Let $C^- = s^{-1}\{1\}$. For every $A^- \subset C^-$, either $p(A^-, C^- - A^-)$ is not a function, or there are conditions $q(A^-) \in \mathbf{P}$ and $r(A^-) \in T$ such that $q(A^-) \leq p(A^-, C^- - A^-), r(A^-)$. Denote $\bigcup\{\mathrm{dom}(q(A^-)) : A^- \subset C^-\} \cap B_\xi = E$. Since B_ξ is almost disjoint from every A_η and D_η for $\eta < \xi$, the set E is finite. Let $t \supset s$ be such that $E \subset \mathrm{dom}(t)$, and $t(n) = 0$ for every $n \in E - \mathrm{dom}(s)$. Note that t does the job: If the characteristic function of C extends t, and $A \dot{\cup} D = C$, then $p(A, D)$ is compatible with $q(A^-)$, and hence, with $r(A^-)$. □

Now, if $T \in V_\beta^+$ and $\tilde{p} \in V_\beta^+$, then also the set $G(\tilde{p}, T)$, which is defined by a Π_1^1-formula with parameters from V_β^+, is in V_β^+. Since C is a Cohen real relative to B_ξ, for every $\tilde{p}$, D as above, the set $C \in G(\tilde{p}, T)$. This means that if $A_\xi \dot{\cup} D_\xi = C$, and $T \in V_\beta^+$, then T remains predense in $\mathbf{P}^+$. □

Proof of 4.2. Suppose C is as in the hypothesis of the lemma. For $A \subset C$ denote: $\mathbf{T}(A) = \mathbf{P}(\overrightarrow{A}^{\xi} \cup \{A\}, \overrightarrow{D}^{\xi} \cup \{C - A\}) \times \mathbf{Q}$, and $\mathbf{T} = \mathbf{P}^{\xi} \times \mathbf{Q}$. Let $\{\dot{y}(n) \colon n \in \omega\}$ be the set of all $\mathbf{T}$-names which receive attention at stage ξ.

For $p \in \mathbf{T}$ and $A \subset C$ we denote:

$$p \vee Ch_A = \langle (p)_0 \cup 1_{A-(p)_0^{-1}\{0\}} \cup 0_{C-(A\cup(p)_0^{-1}\{1\})}, (p)_1 \rangle.$$

Notice that the set $\{p \vee CH_A \colon p \in \mathbf{T}\}$ is dense in $\mathbf{T}(A)$ for every $A \subset C$. For $n \in \omega$ and $p \in \mathbf{T}$, let $A_{n,p} = \{A \subset C \colon p \vee Ch_A \Vdash_{\mathbf{T}(A)} \text{‘}(F(C) \cap \dot{y}(n)) \Delta F(A) \in \mathscr{J}\text{’}\}$ and $\Gamma_{n,p} = \{\langle A, E\rangle \colon A \subset C \& E \subset \omega \& p \vee Ch_A \Vdash_{\mathbf{T}(A)} \text{‘}(F(C) \cap \dot{y}(n)) \Delta E \in \mathscr{J}\text{’}\}$.

4.4 *Claim.* For every $n \in \omega$ and $p \in \mathbf{T}$, the set $\Gamma_{n,p}$ is an analytic subset of $\mathscr{P}(C) \times \mathscr{P}(\omega)$. □

The formal proof of 4.4 is completely straightforward, though tedious. It was expounded in [J1]. The interested reader should have no difficulty proving the claim him—or herself.

Now suppose the lemma does not hold. In our new terminology this means that

$$\bigcup\{A_{n,p} \colon n \in \omega, p \in \mathbf{T}\} = \mathscr{P}(C). \tag{1}$$

Let $P\Gamma_{n,p} = \{A \subset C \colon \exists E \subset \omega \langle A, E\rangle \in \Gamma_{n,p}\}$. By (1),

$$\bigcup\{P\Gamma_{n,p} \colon n \in \omega, p \in \mathbf{T}\} = \mathscr{P}(C). \tag{2}$$

By Claim 4.4, for every n, p the set $P\Gamma_{n,p} \in \Sigma_1^1$; hence, it has the Baire property. By (2), there must be a pair $\langle n, p\rangle$ such that $P\Gamma_{n,p}$ is of second Baire category. We fix such a pair $\langle n, p'\rangle$. Let U_s be a basic subset of $\mathscr{P}(C)$ such that $U_s - P\Gamma_{n,p'}$ is of first Baire category, and suppose $\operatorname{dom}(s) \subset N$. Fix $p \leq p'$ such that $N \subset \operatorname{dom}((p)_0)$, and for every $k \in \operatorname{dom}(s) - \operatorname{dom}((p')_0)$ we have $(p)_0(k) = s(k)$. Notice that $P\Gamma_{n,p}$ is a comeagre subset of $\mathscr{P}(C)$.

By a theorem of J. von Neumann (see [M, p. 240, 4E.9]) there exists a Baire measurable function $F_1 \colon \mathscr{P}(C) \to \mathscr{P}(\omega)$ such that $\langle A, F_1(A)\rangle \in \Gamma_{n,p}$ for all $A \in P\Gamma_{n,p}$.

4.5 **Fact.** Let $\langle A, E\rangle \in \Gamma_{n,p}$. Then $\forall E_1 \subset \omega \langle A, E_1\rangle \in \Gamma_{n,p} \leftrightarrow E \Delta E_1 \in \mathscr{J}$.

Proof. If

$$p \vee Ch_A \Vdash \text{‘}(\dot{y}(n) \cap F(C)) \Delta E \in \mathscr{J}\text{’}$$

and

$$p \vee Ch_A \Vdash \text{‘}(\dot{y}(n) \cap F(C)) \Delta E_1 \in \mathscr{J}, \text{’}$$

then $p \vee Ch_A \Vdash$ ‘$E \Delta E_1 \in \mathscr{J}$.’ By absoluteness of Σ_1^1-formulas, $E \Delta E_1 \in \mathscr{J}$. □

It follows that F_1 is a Baire measurable function $\mathscr{J}$-equivalent to F on a comeagre subset of $\mathscr{P}(C)$. By Lemma 1.2, this implies that $F \restriction \mathscr{P}(C)$ is semitrivial, contradicting the hypothesis of Lemma 4.2. □

5. A PRESERVATION LEMMA

This section is devoted to the proof of Lemma 5.1, which in turn is crucial for the proof of Theorem B in the next section. By $\mathbf{Fn}(\beta, 2)$ we denote the set of all finite partial functions from β into 2 partially ordered by reverse inclusion. We say that a formula is Σ^1, if it is Σ^1_n for some n. In particular, all Π^1_n-formulas are Σ^1. In this and the following section we call a forcing notion $\mathbf{R}$ *uncountable*, if the restriction $\mathbf{R} \restriction r$ is uncountable for every $r \in \mathbf{R}$. I hope that this abuse of terminology does not cause misunderstandings.

5.1 **Lemma.** *Let $\phi(a_1, \dots, a_k)$ be a Σ^1-formula with all parameters shown.*

(a) *If $\mathbf{P}_0$ and $\mathbf{P}_1$ are both uncountable harmless forcing notions, and $a_1, \dots, a_k$ are reals in the ground model, then $\Vdash_{\mathbf{P}_0} \phi(\check{a}_1, \dots, \check{a}_k)$ iff $\Vdash_{\mathbf{P}_1} \phi(\check{a}_1, \dots, \check{a}_k)$.*

(b) *Suppose $\mathbf{R}, \mathbf{R}^+$ are forcing notions such that: $\mathbf{R} \ll_V \mathbf{R}^+$, and $\Vdash_{\mathbf{R}}$ '$\mathbf{R}^+/\mathbf{R}$ is harmless.'*

Suppose furthermore that $\dot{\mathbf{P}}$ is a $\mathbf{R}$-name for a countable forcing, and $\dot{y}_1, \dots, \dot{y}_k$ are $\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)$-names for reals. Then $\Vdash_{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)} \phi(\dot{y}_1, \dots, \dot{y}_k)$ iff $\Vdash_{\mathbf{R}^+ * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)} \phi(\dot{y}_1, \dots, \dot{y}_k)$.

(c) *Suppose $\mathbf{R}, \mathbf{R}^+, \dot{\mathbf{P}}, \dot{y}_1, \dots, \dot{y}_k$ are as in* (b), *and assume that $\dot{\mathbf{P}}^+$ is an $\mathbf{R}^+$-name for a forcing notion such that $\Vdash_{\mathbf{R}^+} \dot{\mathbf{P}} \ll_{V^{\mathbf{R}}} \dot{\mathbf{P}}^+$, and $\Vdash_{\mathbf{R}}$ '$(\mathbf{R}^+ * \dot{\mathbf{P}}^+)/\mathbf{R}$ is harmless.' Then $\Vdash_{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)} \phi(\dot{y}_1, \dots, \dot{y}_k)$ iff $\Vdash_{\mathbf{R}^+ * \dot{\mathbf{P}}^+ \times \mathbf{Fn}(\omega_1, 2)} \phi(\dot{y}_1, \dots, \dot{y}_k)$.*

Although (b) and (c) of the lemma look fiercely technical, their proofs are quite harmless, once (a) is established. They rely on the following claim.

5.2 *Claim.* Let $\mathbf{R}, \mathbf{R}^+, \dot{\mathbf{P}}, \dot{\mathbf{P}}^+$ be as in 5.1. Let $\beta < \omega_1$.

(a) $\Vdash_{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2)}$ '$(\mathbf{R}^+ * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2))/(\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2))$ is harmless.'

(b) $\Vdash_{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2)}$ '$(\mathbf{R}^+ * \dot{\mathbf{P}}^+ \times \mathbf{Fn}(\omega_1, 2))/(\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2))$ is harmless.'

To see how the claim together with 5.1(a) implies 5.1(b),(c), notice first that $\dot{y}_1, \dots, \dot{y}_k$ must in fact be $\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2)$-names for some $\beta < \omega_1$. By repeated applications of Lemma 1.1, we see that $\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2) \ll_V \mathbf{R}^+ * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)$ and $\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2) \ll_V \mathbf{R}^+ * \dot{\mathbf{P}}^+ \times \mathbf{Fn}(\omega_1, 2)$. Therefore, $\dot{y}_1, \dots, \dot{y}_k$ remain names for reals in the languages of the larger forcing notions. Clearly, in $V^{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2)}$, the remainder $(\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2))/(\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2))$ is isomorphic to $\mathbf{Fn}(\omega_1, 2)$, and is therefore an uncountable, harmless forcing notion. The other two remainders are harmless by 5.2, and obviously uncountable. Now apply 5.1(a) in $V^{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2)}$ to $\phi(\dot{y}_1, \dots, \dot{y}_k)$ for the forcing relations $\Vdash_{(\mathbf{R}^+ * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2))/(\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2))}$ resp. $\Vdash_{(\mathbf{R}^+ * \dot{\mathbf{P}}^+ \times \mathbf{Fn}(\omega_1, 2))/(\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\beta, 2))}$.

The following fact is crucial for the proofs in this section.

5.3 **Sublemma.** *Let $\mathbf{R}$ be a harmless notion of forcing, and let $\mathbf{P}$ be countable such that $\mathbf{P} \ll_V \mathbf{R}$. Then $\Vdash_{\mathbf{P}}$ '$\mathbf{R}/\mathbf{P}$ is harmless.'*

We need to recall an alternative characterization of the $\ll_V$-relation.

5.4 **Definition.** Let $\mathbf{P}, \mathbf{R}$ be such that $\mathbf{P} \ll \mathbf{R}$. Call a condition $p \in \mathbf{P}$ a *retraction* of $r \in \mathbf{R}$ to $\mathbf{P}$ iff $\forall q \in \mathbf{P}(q \not\perp p \to q \not\perp r)$.

5.5 *Claim* (Folklore). Suppose $\mathbf{P} \ll \mathbf{R}$. Then $\mathbf{P} \ll_V \mathbf{R}$ iff every condition $r \in \mathbf{R}$ has a retraction to $\mathbf{P}$.

Proof. If $\mathscr{A} \subseteq \mathbf{P}$ is predense in $\mathbf{P}$, and $r \in \mathbf{R}$, then let p be a retraction of r to $\mathbf{P}$. There is $q \in \mathscr{A}$ such that $q \not\perp p$. By 5.4, $q \not\perp r$, so p does not contradict the predensity of $\mathscr{A}$ in $\mathbf{R}$. Now let $r \in \mathbf{R}$, and let $\mathscr{A} \subseteq \mathbf{P}$ be an antichain maximal with respect to the property that if $q \in \mathscr{A}$, then $q \perp r$. If

$\mathbf{P} \ll_V \mathbf{R}$, then $\mathscr{A}$ is not maximal in $\mathbf{P}$. Let $p \in \mathbf{P}$ be such that $q \perp p$ for all $q \in \mathscr{A}$. If $s \in \mathbf{P}$ is compatible with p, and $t \leq_{\mathbf{P}} s, p$, then $t \perp q$ for every $q \in \mathscr{A}$, hence $t \not\perp r$ by the definition of $\mathscr{A}$. This proves that p is a retraction of r to $\mathbf{P}$. □

5.6 **Corollary.** *Assume that $V \subseteq V^+$ are both transitive ε-models. If $V \Vdash \mathbf{P} \ll_V \mathbf{R}$, then $V^+ \Vdash \mathbf{P} \ll_{V^+} \mathbf{R}$.*

Proof. Note that "p is a retraction of r to $\mathbf{P}$" is absolute for transitive ε-models. □

The terminology of retractions provides us with a convenient approach to remainders. Suppose $\mathbf{P} \ll_V \mathbf{R}$, and let $\mathbf{G} \subseteq \mathbf{P}$ be generic over V. Then we identify $\mathbf{R}/\mathbf{P}$ with the set $\{r \in \mathbf{R}: \exists p \in \mathbf{G} p$ is a retraction of r to $\mathbf{P}\}$, partially ordered by the relation: $r_1 \leq_{\mathbf{R}/\mathbf{P}} r$ iff $\forall r_2 \leq_{\mathbf{R}} r_1$ $(r_2 \perp_{\mathbf{R}} r \to \exists p \in \mathbf{G} r_2 \perp_{\mathbf{R}} p)$ iff $\forall r_2 \leq_{\mathbf{R}} r_1$ $(r_2 \perp_{\mathbf{R}} r \to r_2 \notin \mathbf{R}/\mathbf{P})$. Clearly, $p \Vdash_{\mathbf{P}}$ '$r \in \mathbf{R}/\mathbf{P}$' iff p is a retraction of r to $\mathbf{P}$. Notice also that $r \in \mathbf{R}/\mathbf{P}$ iff $\forall p \in \mathbf{G} r \not\perp p$, and that the relation $\leq_{\mathbf{R}/\mathbf{P}}$ contains $\leq_{\mathbf{R}} \cap (\mathbf{R}/\mathbf{P})^2$ as a subset. The following observation is also helpful.

5.7 *Claim.* If $r \not\perp_{\mathbf{R}/\mathbf{P}} r^*$, then there is an $r_1 \in \mathbf{R}/\mathbf{P}$ such that $r_1 \leq_{\mathbf{R}} r, r^*$.

Proof. If $r \not\perp_{\mathbf{R}/\mathbf{P}} r^*$, then there are $p \in \mathbf{G}$ and $r_2 \in \mathbf{R}$ such that $p \Vdash_{\mathbf{P}}$ '$r, r_2, r^* \in \mathbf{R}/\mathbf{P} \,\&\, r_2 \leq_{\mathbf{R}/\mathbf{P}} r, r^*$.' So p is a retraction of r, r^*, r_2. Let $\mathscr{A}$ be a maximal antichain below r_2 in $\mathbf{R}$ such that $q \leq p$ or $q \perp p$ for all $q \in \mathscr{A}$. Since the set of retractions of elements of $\mathscr{A}$ is predense below p, (otherwise p would not be a retraction of r_2), there is some $r_1 \in \mathscr{A} \cap \mathbf{R}/\mathbf{P}$. This r_1 can not be incompatible with p, therefore $r_1 \leq_{\mathbf{R}} p, r_2$. If $r_0 \leq_{\mathbf{R}} r_1$, and if p_0 is a retraction of r_0 to $\mathbf{P}$, then $p_0 \leq_{\mathbf{P}} p$. Therefore, $p_0 \Vdash_{\mathbf{P}}$ '$r, r^*, r_0 \in \mathbf{R}/\mathbf{P} \,\&\, r_0 \leq_{\mathbf{R}/\mathbf{P}} r, r^*$.' By the definition of $\leq_{\mathbf{R}/\mathbf{P}}$, this implies that $r_0 \not\perp_{\mathbf{R}} r$ and $r_0 \not\perp_{\mathbf{R}} r^*$. Since the above is true for all $r_0 \leq_{\mathbf{R}} r_1$, we must have $r_1 \leq_{\mathbf{R}} r, r^*$. □

Proof of 5.2. (a) Since no condition of $\mathbf{R}^+/\mathbf{R}$ is mentioned in the construction of $\dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)$, we can identify $\mathbf{R}^+ * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)$ with $(\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)) * (\mathbf{R}^+/\mathbf{R})$. It remains to show that

(1) $V^{\mathbf{R}} \Vdash$ '$\mathbf{R}^+/\mathbf{R}$ is harmless' implies

(2) $V^{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)} \Vdash$ '$\mathbf{R}^+/\mathbf{R}$ is harmless.'

Since $\dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)$ has precaliber ω_1, and forcing notions of precaliber ω_1 do not destroy the c.c.c., the remainder $\mathbf{R}^+/\mathbf{R}$ retains the c.c.c. in $V^{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)}$. Now let $\mathbf{Q}^- \in V^{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)}$ be a countable subset of $\mathbf{R}^+/\mathbf{R}$. By the c.c.c. of $\mathbf{P} \times \mathbf{Fn}(\omega_1, 2)$, there is a countable $\mathbf{Q} \in V^{\mathbf{R}}$ such that $\mathbf{Q}^- \subseteq \mathbf{Q} \subseteq \mathbf{R}^+/\mathbf{R}$. In $V^{\mathbf{R}}$, we find a countable $\mathbf{Q}^+$ such that $\mathbf{Q} \subseteq \mathbf{Q}^+ \ll_{V^{\mathbf{R}}} \mathbf{R}^+/\mathbf{R}$. By 5.6, $\mathbf{Q}^+ \ll_{V^{\mathbf{R} * \dot{\mathbf{P}} \times \mathbf{Fn}(\omega_1, 2)}} \mathbf{R}^+/\mathbf{R}$. This proves (2).

(b) First we want to convince ourselves that

(3) $\Vdash_{\mathbf{R}^+}$ '$\dot{\mathbf{P}} \ll_{V^{\mathbf{R}}} \dot{\mathbf{P}}^+$' implies

(4) $\Vdash_{\mathbf{R}}$ '$\dot{\mathbf{P}} \ll_{V^{\mathbf{R}}} (\mathbf{R}^+ * \dot{\mathbf{P}}^+)/\mathbf{R}$.'

(4) should be understood in the following way: We identify $\mathbf{P}$ with the set of all pairs $\langle 1_{\mathbf{R}^+}, p\rangle$, where $p \in \mathbf{P}$, and $(\mathbf{R}^+ * \dot{\mathbf{P}}^+)/\mathbf{R}$ with the set of all pairs $\langle r, p\rangle$, where $r \in \mathbf{R}^+$ has some retraction r_0 to $\mathbf{R}$ which is in the generic filter $\mathbf{G} \subset \mathbf{R}$.

Suppose now that (3) does not imply (4). Then there exist: an **R**-name $\dot{D}$ for a predense subset of **P**, and conditions $\langle r, p\rangle \in \mathbf{R}^+ * \dot{\mathbf{P}}^+$ and $r_0 \in \mathbf{R}$ such that $r_0 \Vdash_{\mathbf{R}}$ '$\langle r, p\rangle \in \mathbf{R}^+ * \dot{\mathbf{P}}^+/\mathbf{R}$ and $\langle r, p\rangle \perp q$ for all $q \in \dot{D}$.'

Then r_0 is a retraction of r to **R**. In particular, $r \not\perp_{\mathbf{R}^+} r_0$. Let $r_1 \leq_{\mathbf{R}^+} r, r_0$. Then $r_1 \Vdash_{\mathbf{R}^+}$ '$\forall q \in \dot{D} q \perp_{\dot{\mathbf{P}}^+} p$.' But this contradicts (3).

Since $V^{\mathbf{R}} \Vdash$ '$(\mathbf{R}^+ * \dot{\mathbf{P}}^+)/\mathbf{R}$ is harmless,' by 5.3, also $V^{\mathbf{R}*\dot{\mathbf{P}}} \Vdash$ '$(\mathbf{R}^+ * \dot{\mathbf{P}}^+)/(\mathbf{R}*\mathbf{P})$ is harmless.' Reasoning as in point (a), one easily proves that also $V^{\mathbf{R}*\dot{\mathbf{P}}\times \mathbf{Fn}(\beta, 2)}$ $\Vdash$ '$((\mathbf{R}^+ * \dot{\mathbf{P}}^+)/(\mathbf{R}*\mathbf{P})) \times \mathbf{Fn}(\omega_1, 2)$ is harmless.' Since $\mathbf{Fn}(\omega_1, 2)/\mathbf{Fn}(\beta, 2)$ is isomorphic to $\mathbf{Fn}(\omega_1, 2)$, this implies 5.2(b). □

Proof of 5.3. By 1.1(e), **R** is equivalent to $\mathbf{P} * (\dot{\mathbf{R}}/\mathbf{P})$. Since **R** satisfies the c.c.c., both **P** and **R/P** satisfy the c.c.c. as well (the latter in $V^{\mathbf{P}}$). For **P** this is obvious, for **R/P** not hard to see: Suppose $\dot{\mathscr{A}}$ is a **P**-name and $p \in \mathbf{P}$ is such that

(5) $p \Vdash$ '$\dot{\mathscr{A}}$ is an uncountable antichain in **R/P**.' Consider an uncountable collection $\{\langle p_\xi, r_\xi\rangle : \xi < \omega_1\} \subset \mathbf{P} * \mathbf{R}/\mathbf{P}$ such that $p_\xi \in \mathbf{P}$, $p_\xi \leq p$, $p_\xi \Vdash_{\mathbf{P}}$ 'r_ξ is the ξth element of $\dot{\mathscr{A}}$.'

By the c.c.c. for **R**, we have $\langle p_\xi, r_\xi\rangle \not\perp_{\mathbf{R}} \langle p_\eta, r_\eta\rangle$ for some $\xi < \eta$. This means that there is a $\bar{p} \leq p_\xi, p_\eta$ such that $\bar{p} \Vdash_{\mathbf{P}}$ '$r_\xi \not\perp_{\mathbf{R}/\mathbf{P}} r_\eta$. The latter contradicts (5).

It remains to check the second condition for harmlessness. Assume that in $V^{\mathbf{P}}$, $\mathbf{Q}^-$ is a countable subset of **R/P**. By the c.c.c. of **P**, there is a countable **Q** in V such that $\mathbf{Q}^- \subseteq \mathbf{Q} \subseteq \mathbf{R}$. Since **R** is harmless, there is a countable $\mathbf{Q}^+$ in V such that $\mathbf{P} \cup \mathbf{Q} \subseteq \mathbf{Q}^+ \ll_V \mathbf{R}$.

We show that $\mathbf{Q}^+/\mathbf{P} \ll_{V^{\mathbf{P}}} \mathbf{R}/\mathbf{P}$. If $q, q^* \in \mathbf{Q}^+/\mathbf{P}$, and $q \not\perp_{\mathbf{R}/\mathbf{P}} q^*$, let $r \in \mathbf{R}/\mathbf{P}$ be such that $r \leq_{\mathbf{R}} q, q^*$. Such an r exists by 5.7. Consider an antichain $\mathscr{A} \subseteq \mathbf{R}$ maximal below r and consisting of retractions of r to $\mathbf{Q}^+$, and denote for every $s \in \mathscr{A}$: $\mathrm{re}(s) = \{p : p$ is a retraction of s to $\mathbf{P}\}$. Since $\bigcup\{\mathrm{re}(s) : s \in \mathscr{A}\}$ is predense below r, there must be some retraction q_0 in $\mathscr{A}$ which is simultaneously in $\mathbf{Q}^+/\mathbf{P}$. Clearly, $q_0 \leq q, q^*$. We have thus shown that $q \not\perp_{\mathbf{Q}^+/\mathbf{P}} q^*$, and hence, that $\mathbf{Q}^+/\mathbf{P} \ll \mathbf{R}/\mathbf{P}$.

Suppose now that $r \in \mathbf{R}/\mathbf{P}$. Reasoning as above, we find a retraction q of r to $\mathbf{Q}^+$ so that $q \in \mathbf{Q}^+/\mathbf{P}$. It remains to show that q is a retraction of r to $\mathbf{Q}^+/\mathbf{P}$ as well. If not, there is a $p \in \mathbf{G}$ and a $q_1 \in \mathbf{Q}^+$ such that

(6) $\Vdash_{\mathbf{P}}$ '$q, q_1, r \in \mathbf{R}/\mathbf{P} \,\&\, q \not\perp_{\mathbf{Q}^+/\mathbf{P}} q_1 \,\&\, q_1 \perp_{\mathbf{R}/\mathbf{P}} r$.'

By 5.7, we may without loss of generality assume that $q_1 \leq_{\mathbf{Q}^+} q$. Then q_1 is a retraction of r to $\mathbf{Q}^+$, and p is a common retraction of r and q_1. Let $q_2 \leq_{\mathbf{Q}^+} q_1, p$. Then q_2 is compatible with r, so suppose $r_1 \leq_{\mathbf{R}} q_2, r$. Let p_1 be a retraction of r_1 to **P**. Then p_1 is a retraction of q_1 to **P** as well, and we may also assume that $p_1 \leq_{\mathbf{P}} p$. But then $p_1 \Vdash_{\mathbf{P}}$ '$r_1 \in \mathbf{R}/\mathbf{P}$,' so $p_1 \Vdash_{\mathbf{P}}$ '$q_1 \not\perp_{\mathbf{R}/\mathbf{P}} r$,' which contradicts (6). □

We conclude this section with the

Proof of 5.1(a). If $\mathbf{Q}_0 \ll_V \mathbf{P}_0$ and $\mathbf{Q}_1 \ll_V \mathbf{P}_1$ are both countable, then the Boolean completions of $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are isomorphic. This isomorphism induces an isomorphism $F: V^{\mathbf{Q}_0} \to V^{\mathbf{Q}_1}$ of the classes of $\mathbf{Q}_0$- and $\mathbf{Q}_1$-names. (The reader is advised to interpret the preceding sentence in his or her favored approach to forcing before reading the rest of this proof. If $V^{\mathbf{Q}_0}$, $V^{\mathbf{Q}_1}$ are treated as Boolean-valued models, then they are simply the same. But in what follows, we specifically need a function that sends $\mathbf{Q}_0$-names to $\mathbf{Q}_1$-names. It does not need to be one-to-one though; it suffices that F respects equivalence of names.)

It will be easier to prove the lemma, if we strengthen it a little.

Lemma 5.1(a)′. *If* $\mathbf{P}_0, \mathbf{P}_1$ *are as in* (a) *of Lemma* 5.1, $\mathbf{Q}_0, \mathbf{Q}_1, F$ *as above,* $\varphi(a_1, \dots, a_k, a_{k+1}, \dots, a_{k+m})$ *is a* Σ^1*-formula with all parameters shown,* $a_1, \dots, a_k$ *are reals in the ground model,* $\dot{a}_{k+1}, \dots, \dot{a}_m$ *are* $\mathbf{Q}_0$*-names, then*

$$\Vdash_{\mathbf{P}_0} \phi(\check{a}_1, \dots, \check{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m}) \quad \textit{iff} \quad \Vdash_{\mathbf{P}_1} \phi(\check{a}_1, \dots, \check{a}_k, F(\dot{a}_{k+1}), \dots, F(\dot{a}_{k+m})).$$

We prove 5.1(a)′ by induction over the class of ϕ. If $\phi \in \Sigma^1_2$ (or Π^1_2), then by Shoenfield's Lemma,

$$\Vdash_{\mathbf{P}_0} \phi(\check{a}_1, \dots, \check{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m}) \quad \text{iff} \quad \Vdash_{\mathbf{Q}_0} \phi(\check{a}_1, \dots, \check{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m}),$$

and

$$\Vdash_{\mathbf{P}_1} \phi(\check{a}_1, \dots, \check{a}_k, F(\dot{a}_{k+1}), \dots, F(\dot{a}_{k+m})) \quad \text{iff} \quad \Vdash_{\mathbf{Q}_1} \phi(\check{a}_1, \dots, \check{a}_k, F(\dot{a}_{k+1}), \dots, F(\dot{a}_{k+m})).$$

It is an immediate consequence of the choice of F that

$$\Vdash_{\mathbf{Q}_0} \phi(\check{a}_1, \dots, \check{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m}) \quad \text{iff} \quad \Vdash_{\mathbf{Q}_1} \phi(\check{a}_1, \dots, \check{a}_k, F(\dot{a}_{k+1}), \dots, F(\dot{a}_{k+m})).$$

5.1(a)′ is thus true for $\phi \in \Sigma^1_2 \cup \Pi^1_2$.

Now we assume inductively that the lemma is true for all Π^1_n-formulas ϕ, all $\mathbf{P}_0, \mathbf{P}_1, \mathbf{Q}_0, \mathbf{Q}_1, F$ as above, and suppose

$$\Vdash_{\mathbf{P}_0} \exists x \phi(\check{a}_0, \dots, \check{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m-1}, x).$$

Suppose moreover that all names $\dot{a}_{k+1}, \dots, \dot{a}_{k+m-1}$ are $\mathbf{Q}_0$-names. We let $\mathbf{G} \subseteq \mathbf{Q}_0$ be V-generic and work in $V[\mathbf{G}]$ (which is, n.b., the same transitive class as $V[F(\mathbf{G})]$). There is a $\mathbf{P}_0/\mathbf{Q}_0$-name $\dot{a}_{k+m}$ such that

$$\Vdash^{V[\mathbf{G}]}_{\mathbf{P}_0/\mathbf{Q}_0} \phi(\check{a}_1, \dots, \check{a}_k, \check{a}_{k+1}, \dots, \dot{a}_{k+m}).$$

Since $\mathbf{P}_0/\mathbf{Q}_0$ is harmless, there is in $V[\mathbf{G}]$ a countable $\mathbf{Q}^0 \ll_{V[\mathbf{G}]} \mathbf{P}_0/\mathbf{Q}_0$ such that $\dot{a}_{k+m}$ is already a $\mathbf{Q}^0$-name. Now let $\mathbf{Q}^1$ be countable and such that $\mathbf{Q}^1 \ll_{V[\mathbf{G}]} \mathbf{P}_1/\mathbf{Q}_1$, and fix an isomorphism $G\colon V[\mathbf{G}]^{\mathbf{Q}^0} \to V[F(\mathbf{G})]^{\mathbf{Q}^1}$. Then $\Vdash^{V[F(\mathbf{G})]}_{\mathbf{P}_1/\mathbf{Q}_1} \phi(\check{a}_1, \dots, \check{a}_k, F(\check{a}_{k+1}), \dots, F(\check{a}_{k+m-1}), G(\dot{a}_{k+m}))$. Therefore, $\Vdash^{V[F(\mathbf{G})]}_{\mathbf{P}_1/\mathbf{Q}_1} \exists x \phi(\check{a}_1, \dots, \check{a}_k, F(\check{a}_{k+1}), \dots, F(\check{a}_{k+m-1}), x)$. In other words:

$$\Vdash^{V}_{\mathbf{P}_1} \exists x \phi(\check{a}_1, \dots, \check{a}_k, F(\dot{a}_{k+1}), \dots, F(\dot{a}_{k+m-1}))).$$

We have thus shown that 5.1(a)′ holds for all Σ^1_{n+1}-formulas. For a Π^1_{n+1}-formula $\phi(a_1, \dots, a_k, a_{k+1}, \dots, a_{k+m})$, suppose that

$$\Vdash_{\mathbf{P}_0} \phi(\check{a}_1, \dots, \check{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m})$$

and

$$\nVdash_{\mathbf{P}_1} \phi(\check{a}_1, \dots, \check{a}_k, F(\dot{a}_{k+1}), \dots, F(\dot{a}_{k+m})),$$

i.e., there is some $p \in \mathbf{P}_1$ such that

$$p \Vdash_{\mathbf{P}_1} \neg\phi(\check{a}_1, \dots, \check{a}_k, F(\dot{a}_{k+1}), \dots, F(\dot{a}_{k+m})).$$

But $\neg\phi$ is a Σ^1_{n+1} formula, and $\mathbf{P}_1 \restriction p$ is, by our convention, just another uncountable harmless forcing, and $F^{-1} \restriction p$ (please define this in your favored way) is just another isomorphism. So we may apply what we have already proved in the other direction and get a contradiction. □

6. Proof of Theorem B

In this section the phrase *projective lifting of* $\mathrm{Bor}/\mathcal{I}$ will designate a Boolean homomorphism $H\colon \mathrm{Bor} \to B^1_\infty$ such that $\mathrm{Ker}(H) = \mathcal{I}$ and $H(X)\Delta X \in \mathcal{I}$ for every $X \in \mathrm{Bor}$. $\mathcal{I}$ stands here either for $\mathcal{L}$—the ideal of null sets, or $\mathcal{K}$—the ideal of meagre subsets of $\mathcal{P}((0, 1))$.

If $\phi(y, a_1, \dots, a_k)$ is a Σ^1-formula of one free variable y with all parameters shown, then we denote: $Y[\phi, a_1, \dots, a_k] = \{y \in (0, 1)\colon \phi(y, a_1, \dots, a_k)\}$. Then $\mathcal{B}^1_\infty$ is the Boolean subalgebra of $\mathcal{P}((0, 1))$ formed by all sets $Y[\phi, a_1, \dots, a_k]$. Similarly as the proof of Theorem A, the proof of Theorem B relies on a Main Lemma.

6.1 **Main Lemma B.** *Suppose* CH *holds and we are given:*

- an ideal $\mathcal{I}$ which is either $\mathcal{L}$ or $\mathcal{K}$;

- a projective lifting $H\colon \mathrm{Bor} \to \mathcal{B}^1_\infty$ of $\mathrm{Bor}/\mathcal{I}$;

- a sequence $\langle V(\xi)\colon \xi < \omega_1\rangle$ of transitive models for a sufficiently large fragment of ZFC *such that*

. $V(\xi) \subset V(\eta)$ for $\xi < \eta < \omega_1$,

. $H(\omega_1) = \bigcup_{\xi<\omega_1} V(\xi) \cap H(\omega_1)$.

. Whenever $D \subset \omega_1$, then the set $ND = \{\xi < \omega_1\colon D \cap \xi \in V_\xi\}$ contains a c.u.b. subset of ω_1;

- a sequence $\langle Z_\xi\colon \xi < \omega_1\rangle$ of subsets of ω such that Z_ξ is a Cohen real over $V(\xi)$ for $\xi < \omega_1$.

Then there exist

- a normal function $g\colon \omega_1 \to \omega_1$;

- a sequence $\langle \mathbf{P}^\xi\colon \xi < \omega_1\rangle$ of countable forcing notions;

- a $\mathbf{P} = \bigcup_{\xi<\omega_1} \mathbf{P}^\xi$-name $\dot{X}$ for an open subset of the interval $(0, 1)$ such that

(T1) $\mathbf{P}^{\xi+1} \in V(g(\xi + 1))$ *for* $\xi < \omega_1$;

(T2) $\mathbf{P}^\xi \lessdot_{V(g(\xi))} \mathbf{P}^\eta$ *for* $\xi < \eta < \omega_1$;

(T3) $|\mathbf{P}| = \omega_1$;

(T4) $\mathbf{P}^\lambda = \bigcup_{\xi<\lambda} \mathbf{P}^\xi$ *for* $\lambda \in \mathrm{Lim} \cap \omega_1$;

(T5) *For every Σ^1-formula $\phi(y, a_1, \dots, a_k)$, $\mathbf{P} \times \mathbf{Fn}(\omega_1, 2)$ names $\dot{a}_1, \dots, \dot{a}_k$ for reals and every $\langle p, q\rangle \in \mathbf{P} \times \mathbf{Fn}(\omega_1, 2)$, there is a cofinal subset $T \subset \omega_1$ so that for all $\eta \in T$, if*

(T5.1) $\langle p, q\rangle \Vdash_{\mathbf{P}^{\eta+1} \times \mathbf{Fn}(\omega_1, 2)}$ '$Y[\phi, \dot{a}_1, \dots, \dot{a}_k]\Delta\dot{X} \in \mathcal{I}$,' *then there are an open set $U \subset (0, 1)$, a real $x \in (0, 1)$, and a condition $\langle p', q'\rangle \le \langle p, q\rangle$ such that:*

(T5.2) *U, $H(U)$ (i.e., the parameters in the Σ^1-formula which defines $H(U)$), $x, \dot{a}_1, \dots, \dot{a}_k$ are all members of $V_{f(g(\eta+1))}$, and*

either

(T5.3) $\langle p', q'\rangle \Vdash_{\mathbf{P}^{\eta+1} \times \mathbf{Fn}(\omega_1, 2)}$ '$\check{x} \in Y[\phi, \dot{a}_1, \dots, \dot{a}_k] \,\&\, \check{U} \cap \dot{X} = \varnothing$,' *or*

(T5.4) $\langle p', q'\rangle \Vdash_{\mathbf{P}^{\eta+1}\times \mathbf{Fn}(\omega_1, 2)}$ '$\check{x} \notin Y[\phi, \dot{a}_1, \dots, \dot{a}_k] \,\&\, \check{U} \subset \dot{X}$.'

Moreover, these objects may be constructed in V *in such a way that the following holds: Suppose there is a richer world* V^+ *which knows of an increasing sequence* $\langle V(\xi)^+ : \xi < \omega_1\rangle$ *of models for a sufficiently large fragment of* ZFC *such that for all* $\xi < \omega_1$;

- $V(\xi) \subseteq V(\xi)^+$,
- Z_ξ *is a Cohen real over* $V(\xi)^+$. *Then the following holds in* V^+:

(T2+) $\mathbf{P}^\xi \lessdot_{V(g(\xi))^+} \mathbf{P}^\eta$ *for* $\xi < \eta < \omega_1$.

One can prove 6.1 by an argument very similar to the one in [Sh1], although its adaptation to the framework of innocuous iterations requires substantial modifications. These, together with some generalizations of the results in this section can be found in [BJ]. For this reason I do not want to duplicate the proof of 6.1 here. □

We show how 6.1 implies Theorem B. The reasoning will strictly parallel that in §3.

6.2 *Claim.* Suppose $\mathbf{P}$ is as in 6.1. Then $\mathbf{P}$ satisfies the c.c.c.

Proof. Like 3.2. □

Let $V, \kappa, \mathcal{J}$ be as in the hypothesis of Theorem B. We construct an innocuous iteration $\mathbf{R}_\kappa$ of forcing notions $\langle \dot{\mathbf{P}}_\alpha : \alpha < \kappa\rangle$. As explained in §3, even before the actual construction of $\mathbf{R}_\kappa$ we may fix a sequence $\langle \dot{H}'_\alpha : \alpha < \kappa\rangle$ of certain potential $\mathbf{R}_\kappa$-names for functions $H\colon \mathrm{Bor} \to \mathcal{B}^1_\infty$ in such a way that for every potential $\mathbf{R}_\kappa$-name $\dot{H}$ for a function from Bor into B^1_∞ the set:

$S(\dot{H}) = \{\alpha \in S : \dot{H} \restriction \mathrm{Bor} \cap V_\alpha = \dot{H}'_\alpha \restriction \mathrm{Bor} \cap V_\alpha\}$ is stationary in κ. By $\dot{H}_\alpha$ we shall denote $\dot{H}'_\alpha \restriction \mathrm{Bor} \cap V_\alpha$.

6.3 *Remark.* Here, as in some other places of this proof, we treat Bor as the set of *codes* for Borel sets, and similarly, we think of elements of $\mathcal{B}^1_\infty$ as represented by Σ^1-formulas together with their sets of parameters (which also can be coded by reals if one likes). I hope that if not explicitly stated, it will always be clear from the context whether a Borel or projective set is to be treated as the actual set or its code.

Now we construct inductively the forcing notions $\dot{\mathbf{P}}_\alpha$. If $\alpha \in \kappa - S$, then $\dot{\mathbf{P}}_\alpha$ is an $\mathbf{R}_\alpha$-name for the Cohen forcing. If $\alpha \in S$, then we check whether the following conditions hold:

(i) $\dot{H}_\alpha$ is an $\mathbf{R}_\alpha$-name for a function from Bor to $\mathcal{B}^1_\infty$;

(ii) Let $\dot{X}_0, \dot{X}_1$ be $\mathbf{R}_\alpha$-names for Borel sets.

(iia) If $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{X}_0 \Delta \dot{X}_1 \in \mathcal{J}$,' then $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{H}_\alpha(\dot{X}_0) = \dot{H}_\alpha(\dot{X}_1)$,'

(iib) $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{H}_\alpha(\dot{X}_0)\Delta \dot{X}_0 \in \mathcal{J}$,'

(iic) $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{H}_\alpha((0, 1) - \dot{X}_0) = (0, 1) - \dot{H}_\alpha(\dot{X}_0)$,' and

(iid) $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{H}_\alpha(\dot{X}_0 \cup \dot{X}_1) = \dot{H}_\alpha(\dot{X}_0) \cup \dot{H}_\alpha(\dot{X}_1)$.'

In point (ii), the $\dot{H}_\alpha(\dot{X})$ are treated as *sets* of reals. E.g., in (iia) we do *not* require that $\dot{H}_\alpha(\dot{X}_0)$ and $\dot{H}_\alpha(\dot{X}_1)$ are necessarily represented by the same code. We may think of the formulas to the right of the "$\Vdash$"-symbols in (ii) as Σ^1-formulas. E.g., if $H_\alpha(X_0) = Y[\phi, a_1, \dots, a_k]$, $H_\alpha(X_1) = Y[\psi, b_1, \dots, b_m]$, then the formula "$H_\alpha(X_0) = H_\alpha(X_1)$" becomes: "$\forall y \phi(y, a_1, \ldots, a_k) \leftrightarrow \psi(y, b_1, \ldots, b_m)$." We should keep in mind that there is a certain shortcut here: In general, there is no single ϕ such that $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{H}_\alpha(\dot{X}) = Y[\phi, a_1, \ldots, a_k]$,' but only a maximal antichain $\mathscr{A} \subset \mathbf{R}_\alpha$ such that for $p \in \mathscr{A}$, we have $p \Vdash_{\mathbf{R}_\alpha}$ '$\dot{H}_\alpha(\dot{X}) = Y[\phi_p, a_1^p, \ldots, a_{k(p)}^p]$.'

If one of these conditions does *not* hold, then $\dot{\mathbf{P}}_\alpha$ will be $\mathbf{Fn}(\omega_1, 2)$. If (i) and (ii) hold, then $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{H}_\alpha$ is a projective lifting of $\mathrm{Bor}/\mathscr{I}$.'

In this situation we want to design $\dot{\mathbf{P}}_\alpha$ so as to destroy the lifting. We fix a function $f_\alpha \colon \omega_1 \to \alpha$ normal in α and such that its range is disjoint from S, and a sequence $\mathscr{Z}^\alpha = \langle Z_\xi^\alpha : \xi < \omega_1 \rangle$ of reals so that in V_α the hypotheses of the Main Lemma B are satisfied with $H_\alpha = H$ and $V(\xi) = V_{f(\xi)}$ for all ξ. Again, if CH holds in V_α, then we let $\dot{\mathbf{P}}_\alpha$ be an $\mathbf{R}_\alpha$-name for a forcing notion $\mathbf{P}$ so that (T1)–(T5) and (T2+) are satisfied in V_α. If CH does not hold in V_α (i.e., if $\alpha > \omega_2$), then we choose an ω-ended structure $V^- \prec_{\Sigma_{100}} V$ of cardinality ω_1 such that $\omega_1 \subset V^-$ and everything relevant is an element of V^-. Denote $\mathbf{R}_\alpha^- = \mathbf{R}_\alpha \cap V^-$.

6.4 *Claim.* $\Vdash_{\mathbf{R}_\alpha}$ '$\dot{H}_\alpha$ is a projective lifting of $\mathrm{Bor}/\mathscr{I}$.'

Proof. This boils down to showing that (i) and (ii) are still satisfied if we replace $\dot{H}_\alpha$ by $\dot{H}_\alpha \cap V^-$ and $\Vdash_{\mathbf{R}_\alpha}$ by $\Vdash_{\mathbf{R}_\alpha^-}$. Recall that all formulas to the right of the $\Vdash_{\mathbf{R}_\alpha}$-symbol in (ii) are Σ^1. Notice further that if $\phi(a_1, \ldots, a_k)$ is Σ^1, and $\Vdash_{\mathbf{R}_\alpha} \phi(\dot{a}_1, \ldots, \dot{a}_k)$, then there is some $\gamma < \alpha$ such that $\dot{a}_1, \ldots, \dot{a}_k$ are $\mathbf{R}_\gamma^-$-names (where $\mathbf{R}_\gamma^- = \mathbf{R}_\gamma \cap V^-$). This follows from the c.c.c. and the fact that $\mathrm{cf}(\alpha) = \omega_1$. Now $\Vdash_{\mathbf{R}_{\gamma+1}^-}$ '$\Vdash_{\mathbf{R}_\alpha/\mathbf{R}_{\gamma+1}^-} \phi(\check{a}_1, \ldots, \check{a}_k)$.'

By 2.11, the remainder $\mathbf{R}_\alpha/\mathbf{R}_{\gamma+1}^-$ is equivalent to an innocuous iteration. Hence, $\Vdash_{\mathbf{R}_{\gamma+1}^-}$ '$\phi(\dot{a}_1, \ldots, \dot{a}_k)$ holds in some extension via an uncountable harmless forcing.' By 5.1(a), $\Vdash_{\mathbf{R}_{\gamma+1}^-}$ '$\phi(\dot{a}_1, \ldots, \dot{a}_k)$ holds in every extension via an uncountable harmless forcing.' By 2.9 and absoluteness of the relevant formulas, and since $\sup V^- \cap \alpha = \alpha > \gamma + 1$, $\Vdash_{\mathbf{R}_{\gamma+1}^-}$ '$\mathbf{R}_{\gamma+1,\alpha}^-$ is equivalent to an innocuous iteration of uncountable length.' It follows that $\Vdash_{\mathbf{R}_\alpha^-} \phi(\dot{a}_1, \ldots, \dot{a}_k)$. □

We have thus shown that $V_\alpha^- = V^{\mathbf{R}_\alpha^-}$ satisfies the hypothesis of the Main Lemma B. We let $\dot{\mathbf{P}}_\alpha$ be an $\mathbf{R}_\alpha^-$-name for a forcing notion $\mathbf{P}$ so that (T1)–(T5) and (T2+) are satisfied in V_α^-. Since the forcing notions $\mathbf{P}^\xi$ are countable, we may without loss of generality assume that the underlying set of $\mathbf{P}^\xi$ is $\omega \times \xi$ for all ξ.

This finishes the description of $\mathbf{R}_\kappa$. We show that it works.

6.5 *Claim.* $\mathbf{R}_\kappa$ is innocuous.

Proof. Like the proof of 3.4. □

6.6 **Corollary.** $\mathbf{R}_\kappa$ *satisfies the c.c.c.* □

6.7 **Corollary.** $V_\kappa \Vdash$ '$2^\omega = \kappa$.' □

It remains to show that in V_κ there is no projective lifting of $\mathrm{Bor}/\mathscr{I}$. Suppose there is one, and let $\dot{H}$ be its name. For simplicity of notation, assume $\Vdash_{\mathbf{R}_\kappa}$ '$\dot{H}$ is a projective lifting of $\mathrm{Bor}/\mathscr{I}$.'

Let $\alpha \in S(\dot{H})$. At stage α of the construction of $\mathbf{R}_\kappa$ both (i) and (ii) were satisfied. For (i) this is obvious. For point (ii), reason as in the proof of 6.4: If $\phi(\dot{a}_1, \dots, \dot{a}_k)$ is one of the formulas to the right of the "$\Vdash$"-symbol in (ii), then all parameters are already in $V_{\gamma+1}$ for some $\gamma < \alpha$. Then $\Vdash_{\mathbf{R}_{\gamma+1}}$ '$\Vdash_{\mathbf{R}_{\gamma+1,\kappa}} \phi(\check{a}_1, \dots, \check{a}_k)$,' hence, $\Vdash_{\mathbf{R}_{\gamma+1}}$ '$\phi(\dot{a}_1, \dots, \dot{a}_k)$ holds in some (and therefore all) extensions via an uncountable harmless forcing,' hence $\Vdash_{\mathbf{R}_{\gamma+1}}$ '$\Vdash_{\mathbf{R}_{\gamma+1,\alpha}} \phi(\check{a}_1, \dots, \check{a}_k)$,' hence $\Vdash_{\mathbf{R}_\alpha} \phi(\dot{a}_1, \dots, \dot{a}_k)$.

That means that we iterate at stage α a forcing notion $\dot{\mathbf{P}}_\alpha$ which satisfies (T1)–(T5) and (T2+) of the Main Lemma B. Consider the open set $X \subset (0, 1)$ whose existence the Main Lemma B postulates. Let $\dot{X}$ be an $\mathbf{R}^-_{\alpha+1}$-name for X, where $\mathbf{R}^-_{\alpha+1} = \mathbf{R}^-_\alpha * \dot{\mathbf{P}}_\alpha$. We show that no $Y[\phi, \dot{a}_1, \dots, \dot{a}_k]$ (where $\dot{a}_1, \dots, \dot{a}_k$ are $\mathbf{R}_\kappa$-names) can be a value for $\dot{H}[\dot{X}]$.

Let $\langle r, p, q\rangle$ be an element of $\mathbf{R}^-_\alpha * \dot{\mathbf{P}}_\alpha \times \mathbf{Fn}(\omega_1, 2)$, and let $\phi(y, a_1, \dots, a_k)$ be a Σ^1-formula with one free variable y and all parameters shown. Furthermore, let $\dot{a}_1, \dots, \dot{a}_k$ be $\mathbf{R}^-_{\alpha+1} \times \mathbf{Fn}(\omega_1, 2)$-names for reals. By the construction of $\dot{\mathbf{P}}_\alpha$, there are $\langle r', p', q'\rangle \in \mathbf{R}^-_\alpha * \dot{\mathbf{P}}^{\eta+1} \times \mathbf{Fn}(\omega_1, 2)$, and $\mathbf{R}^-_{\alpha+1}$-names $\dot{x}$ for a real and $\dot{U}$ for an open set such that either:

(T5.1) $\langle r', p', q'\rangle \Vdash_{\mathbf{R}^-_\alpha * \dot{\mathbf{P}}^{\eta+1} \times \mathbf{Fn}(\omega_1, 2)}$ '$Y[\phi, \dot{a}_1, \dots, \dot{a}_k] \Delta \dot{X} \notin \mathscr{I}$,' or

(T5.3) $\langle r', p', q'\rangle \Vdash_{\mathbf{R}^-_\alpha * \dot{\mathbf{P}}^{\eta+1} \times \mathbf{Fn}(\omega_1, 2)}$ '$\phi(\dot{x}, \dot{a}_1, \dots, \dot{a}_k) \,\&\, \dot{U} \cap \dot{X} = \varnothing \,\&\, \dot{x} \in \dot{H}^-_\alpha[\dot{U}]$,' or

(T5.4) $\langle r', p', q'\rangle \Vdash_{\mathbf{R}^-_\alpha * \dot{\mathbf{P}}^{\eta+1} \times \mathbf{Fn}(\omega_1, 2)}$ '$\neg\phi(\dot{x}, \dot{a}_1, \dots, \dot{a}_k) \,\&\, \dot{U} \subset \dot{X} \,\&\, \dot{x} \in \dot{H}^-_\alpha[\dot{U}]$.'

Notice that in either case the formula to the right of the "$\Vdash$"-symbol is a Σ^1-formula $\psi(\dot{x}, \dot{a}_1, \dots, \dot{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m})$. The fact that a given projective set is null or meagre can be expressed by a Σ^1-formula. The extra parameters come from the definitions of $\dot{U}$ and $\dot{H}^-_\alpha[\dot{U}]$.

We may assume that all the relevant names: $\dot{x}, \dot{a}_1, \dots, \dot{a}_{k+m}, \dot{U}$ are $\mathbf{R}^-_{f(g(\eta+1))} * \dot{\mathbf{P}}^{\eta+1} \times \mathbf{Fn}(\beta, 2)$-names, for some $\beta < \omega_1$. To see this, reason as follows: By the cofinality of T in ω_1, we may choose η large enough so that $\dot{a}_1, \dots, \dot{a}_k$ are elements of $V_{f(g(\eta+1))}$. Condition (T5.2) of the Main Lemma takes care of the other names (in case of clauses (T5.3) or (T5.4)).

Denote $\gamma = f(g(\eta+1))$. By our choice of f, $\mathrm{cf}(\gamma) \neq \omega_1$. Therefore, reasoning as in the proof of 2.11, one can show that in $V^{\mathbf{R}^-_\gamma}$, the remainder $\mathbf{R}^-_\alpha / \mathbf{R}^-_\gamma$ is harmless.

Let r^* be a retraction of r' to $\mathbf{R}^-_\gamma$. Applying Lemma 5.1(b) with $\mathbf{R}^-_\alpha \restriction r'$ in the role of $\mathbf{R}^+$, and $\mathbf{R}^-_\gamma \restriction r^*$ in the role of $\mathbf{R}$, we infer that

$$\langle r^*, p', q'\rangle \Vdash_{\mathbf{R}^-_\gamma * \dot{\mathbf{P}}^{\eta+1} \times \mathbf{Fn}(\omega_1, 2)} \psi(\dot{x}, \dot{a}_1, \dots, \dot{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m}).$$

From 5.1(c) with $\mathbf{R}^-_\gamma$ in the role of $\mathbf{R}$, $\mathbf{R}^-_\alpha$ in the role of $\mathbf{R}^+$, $\dot{\mathbf{P}}^{\eta+1}$ in the role of $\dot{\mathbf{P}}$, and $\dot{\mathbf{P}}$ in the role of $\mathbf{P}^+$, we deduce that $\langle r^*, p', q'\rangle \Vdash_{\mathbf{R}^-_{\alpha+1}\mathbf{Fn}(\omega_1, 2)}$ $\psi(\dot{x}, \dot{a}_1, \dots, \dot{a}_k, \dot{a}_{k+1}, \dots, \dot{a}_{k+m})$.

Because the set of $\langle r', p'\rangle$'s as above is dense below $\langle r, p\rangle$, we have for any particular Σ^1-formula $\phi(y, a_1, \dots, a_k)$:

(*) $\langle r, p\rangle \Vdash_{\mathbf{R}^-_{\alpha+1}}$ 'In any extension of $V^{\mathbf{R}^-_{\alpha+1}}$ via a countable forcing $\mathbf{Q}$, and for any $\mathbf{Q}$-names $\dot{a}_1, \ldots, \dot{a}_k$, there is a Σ^1-formula

$$\psi(\dot{x}, \dot{a}_1, \ldots, \dot{a}_k, \dot{a}_{k+1}, \ldots, \dot{a}_{k+m})$$

which implies that $H^-_\alpha[\dot{X}]$ cannot be $Y[\phi, \dot{a}_1, \ldots, \dot{a}_k]$, and such that $\psi(\dot{x}, \dot{a}_1, \ldots, \dot{a}_k, \dot{a}_{k+1}, \ldots, \dot{a}_{k+m})$ holds in some (and therefore all) uncountable harmless forcing extensions of $V^{\mathbf{R}^-_{\alpha+1}*\mathbf{Q}}$.'

Now recall that by 2.11, $\Vdash_{\mathbf{R}^-_{\alpha+1}}$ '$\mathbf{R}_\kappa/\mathbf{R}^-_{\alpha+1}$ is harmless.'

Let $a_1, \ldots, a_k$ be in V_κ. Then $a_1, \ldots, a_k$ are already present in $V^{\mathbf{R}^-_{\alpha+1}*\mathbf{Q}}$, where $\mathbf{Q}$ is some countable fragment of the remainder $\mathbf{R}_\kappa/\mathbf{R}^-_{\alpha+1}$. By 5.3, $\mathbf{R}_\kappa/(\mathbf{R}^-_{\alpha+1} * \mathbf{Q})$ is an uncountable harmless forcing extension, and now it follows from (*) that $Y[\phi, a_1, \ldots, a_k]$ cannot be $H[X]$ in V_κ. □

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Department of Mathematics, Ohio University, Athens, Ohio 45701